Strongly Semiunits and Tri-Regular Elements in Rings

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Received: October 4, 2018 Accepted: November 21, 2018 Online Published: December 1, 2018

doi: 10.23918/eajse. v4i2p141

Abstract: In this paper we study semiunit elements in the group ring Z_2G, where G is a cyclic group and we introduce and discuss strongly semiunit elements in Z_n, for $n = p, 2p, p^2$, where p is an odd prime. We define and study tri-regular elements in Z_n and in the group ring Z_2G, where G is a cyclic group.

Keywords: Semiunit, Strongly Semiunit and Tri-Regular Element

1. Introduction

Smarandache concepts were first introduced by Florentin Smarandache[9]. In 2002 W. B. Vasantha Kandasamy published a book titled "Smarandache Ring" [10], she introduced Semiunit, Smarandache unit elements. This work consists of three sections. In section one we state basic definitions and some results that we need in our work. In section two we study semiunit elements in Z_n and in the group ring Z_2G, where G is a cyclic group of order n. We introduce and study the concept of Strongly semiunit element in Z_n, for $n = p, 2p, p^2$, where p is an odd prime. In section three we define tri-regular elements in Z_n and in the group ring Z_2G, where G is a cyclic group.

2. Background

In this section we state basic definitions that we need in our work.

Definition 2.1 [10]

Let R be a commutative ring with identity 1. An element x of R is said to be a semiunit of R if there exists $y \in R$ such that $(x + 1)(y + 1) = 1$.

Definition 2.2 [4]

An element α of a ring R is called an m-idempotent, if $\alpha^m = \alpha$. An m-idempotent is said to be a non-trivial m-idempotent if $\alpha^l \neq \alpha$ for each $0 < l < m$. So an element α of a ring R is called tripotent (3-idempotent) if $\alpha^3 = \alpha$. A tripotent element is said to be a non-trivial tripotent if $\alpha^2 \neq \alpha$.

Definition 2.3[10]

Let R be a ring. An element $0 \neq \alpha \in R$ is called a super idempotent of R, if $\alpha^2 - \alpha$ is an idempotent.

Remark 2.4 [5]

Let R be a nontrivial ring with identity and $e \in R$ be an idempotent. Then $1 - e$ is also idempotent.

Proposition 2.5 [6]

In Z_n, if y and $2y + 1$ are non-trivial tripotents and $gcd(n, 12) = 1$, then $y + 1$ is a non-trivial idempotent and $y \neq \frac{n}{2}$(in case n is even).

Definition 2.6 [10]

A ring R is called E-ring if there exists $n \in \mathbb{Z}^+$, $x^{2n} = x$ and $2x = 0$ for every $x \in R$ and the minimal such n is called the degree of the E-ring. It is interesting that an E-ring of degree 2 is a Boolean ring.

Theorem 2.7 [1]

The linear congruence $ax \equiv b(mod\ n)$ has a solution if and only if $d \mid b$, where $d = gcd(a, n)$. If $d \mid b$, then it has exactly d mutually incongruent solutions modulo n.

Definition 2.8 [10]

Let R be a ring with identity. We say $x \in R\backslash\{1\}$ is a Smarandache unit if there exists $y \in R$ with

1. $xy = 1$.
2. There exist $a, b \in R\backslash\{x, y, 1\}$, such that
   $i)$ $xa = y$ or $ax = y$ or
   $ii)$ $yb = x$ or $by = x$ and
   $iii)$ $ab = 1$.

Theorem 2.9 [2]

Let R be a ring with identity and x a unit different from 1. Then x is a Smarandache unit if and only if $x^2 \neq 1$ and $x^3 \neq 1$.

Theorem 2.10 [10]

If F is a prime field of characteristic 0, then every unit is a Smarandache unit.

Definition 2.11 [10]

Let R be a ring. An element $0 \neq x \in R$ is said to be Smarandache idempotent of R if

1. $x^2 = x$.
2. There exists $a \in R\backslash\{0, 1, x\}$ such that
   $i)$ $a^2 = x$ and
   $ii)$ $xa = a(ax = a)$ or $ax = x(xa = x)$.

Proposition 2.12 [3]

If α is a Smarandache idempotent of the group ring Z_2G, where G is a cyclic group of order n, then 1+ α is a Smarandache idempotent of Z_2G.

Lemma 2.13 [6]

Let R be a ring. If α is a non-trivial tripotent such that $\alpha^2 \neq 1$, then α^2 is a non-trivial Smarandache idempotent.

Theorem 2.14 [3]

Let Z_2G be the group ring, where $G = \langle g, g^{2p} = 1 \rangle$ is a cyclic group of order $2p$, p is a Mersenne prime. Then every element of the form $\alpha = g^{2l} + g^{2^2 l} + \cdots + g^{2^{k-1} l} + g^{2^k l}$ is a Smarandache idempotent.

Theorem 2.15 [9]

Let Z_2G be the group ring of finite cyclic group G of order $2^n p$, where p is a prime of the form $2^n t + 1$. Then the group ring Z_2G has non-trivial Smarandache idempotents.

Theorem 2.16 [2]

The group ring Z_nG, where n is an odd number and G is cyclic group of order $n + 1$ generated by g, has at least two non-trivial Smarandache idempotents.

Theorem 2.17 [10]

Let F be a field of characteristic zero and G is any group of finite order. The group ring FG has a Smarandache idempotent.

Definition 2.18 [10]

Let R be a ring. R is said to be a Smarandache P-ring, if R is a Smarandache ring and it has a subring P such that $x^p = x$ and $px = 0$ for every $x \in P$.

3. Semiunits and Strongly Semiunits

In this section we study semiunits in Z_n, we give a condition on the order of a group G under which the group ring Z_2G has semiunit elements, and we give a sufficient condition under which a semiunit of Z_n is a Strongly semiunit for $n \in \{p, 2p, p^2\}$ where p is a prime.

Proposition 3.1

In Z_n with the prime factorization of $n = {p_1}^{\alpha_1}\ {p_2}^{\alpha_2}\ {p_3}^{\alpha_3} \ldots {p_r}^{\alpha_r}$, and $\alpha_i \geq 1$, an element x is a semiunit if and only if x is not of the form $lp_i - 1$, for some $l \in \mathbb{Z}^+$.

Proof. Let $x = lp_i - 1$ for each i. Then $x + 1 = lp_i$ which is not a unit, hence x is not a semiunit.

If $x \neq lp_i - 1$, then $x + 1$ is a unit in Z_n, hence x is a semiunit.

Proposition 3.2

In the group ring Z_2G, where G is a cyclic group of order n generated by g, if x has an odd number of summands, then x is not a semiunit.

Proof. Note that if x has an odd number of summands, then

$(x+1)(1+g+g^2+g^3+\cdots+g^{n-2}+g^{n-1})=0$, so $x+1$ is not a unit, consequently x is not a semiunit.

Remark 3.3

In the group ring Z_2G, where G is a cyclic group of order n generated by g, the element $1+g^k$, for $1 \leq k \leq n-1$, is a semiunit, since $(1+g^k)+1=g^k$ and in this case g^k is a unit.

Remark 3.4

In the group ring Z_2G, where G is a cyclic group of order n generated by g, if n is even, then the element $x=1+g+g^2+g^3+\cdots+g^{n-1}$ is a semiunit, since

$(x+1)^2=g^2+g^4+g^6+\cdots+g^{n-2}+1+g^{n+2}+\cdots+g^{2n-4}+g^{2n-2}=1$.

Theorem 3.5

In the group ring Z_2G, where G is a cyclic group of order $2n$ generated by g, the element $x=g^k+g^{n+k}$ is a semiunit for each positive integer k.

Proof. Take $y=g^k+g^{n+k}$. Then

$$(x+1)(y+1)=\left(g^k+g^{n+k}+1\right)\left(g^k+g^{n+k}+1\right)$$

$$=g^{2k}+g^{2n+2k}+1=1 .$$

Hence x is a semiunit.

Proposition 3.6

In the group ring Z_2G, where G is a cyclic group of order 2^n for $n>1$, generated by g, the element $x=g+g^{2^{n-1}+1}$ is a semiunit.

Proof. Take $y=g+g^{2^{n-1}+1}$. Then

$$(x+1)(y+1)=(g+g^{2^{n-1}+1}+1)(g+g^{2^{n-1}+1}+1)$$

$$=g^2+g^{2.2^{n-1}+2}+1$$

$$=1.$$

Hence x is a semiunit.

Theorem 3.7

In the group ring Z_2G, where G is a cyclic group of order 2^n (n is an odd integer) generated by g, the two elements $g + g^2$ and $g + g^{2^n-1}$ are semiunits.

Proof. Let $x = g + g^2$, and take

$$y = 1 + g + g^2 + g^4 + g^5 + g^7 + g^8 + \cdots + g^{2^n-3} + g^{2^n-2} + g^{2^n-1}$$

$$= 1 + \sum_{i=0 \text{ with } 3\nmid i}^{2^n-1} g^i.$$

We have to show that $(x + 1)(y + 1) = 1$, for this purpose we describe the multiplication $(x + 1)(y + 1)$ in the following array :

$$A := \begin{pmatrix} \boxed{g} & g^2 & \boxed{g^4} & g^5 & \boxed{g^7} & g^8 & \cdots & \boxed{g^{2^n-6}} & \boxed{g^{2^n-4}} & g^{2^n-3} & \boxed{g^{2^n-1}} \\ g^2 & g^3 & g^5 & g^6 & g^8 & g^9 & \cdots & g^{2^n-5} & g^{2^n-3} & g^{2^n-2} & 1 \\ g^3 & \boxed{g^4} & g^6 & \boxed{g^7} & g^9 & \boxed{g^{10}} & \cdots & \boxed{g^{2^n-4}} & g^{2^n-2} & \boxed{g^{2^n-1}} & \boxed{g} \end{pmatrix}$$

Hence $(x + 1)(y + 1) = 1$ and x is a semiunit.

To show that $x = g + g^{2^n-1}$ is a semiunit, take

$$y = g^2 + g^3 + g^5 + g^6 + g^8 + g^9 + \cdots + g^{2^n-5} + g^{2^n-3} + g^{2^n-2}$$

$$= \sum_{k=2 \text{ and } 3\nmid k-1}^{2^n-2} g^k.$$

We have to show that $(x + 1)(y + 1) = 1$, for this purpose we also describe the multiplication $(x + 1)(y + 1)$ in the following array say A :

$$A = \begin{pmatrix} 1 & g^2 & g^3 & g^5 & g^6 & \cdots & \boxed{g^{2^n-5}} & \boxed{g^{2^n-3}} & g^{2^n-2} \\ g & g^3 & g^4 & g^6 & \boxed{g^7} & \cdots & g^{2^n-4} & g^{2^n-2} & g^{2^n-1} \\ \boxed{g^{2^n-1}} & g & g^2 & g^4 & g^5 & \cdots & \boxed{g^{2^n-6}} & g^{2^n-4} & g^{2^n-3} \end{pmatrix}$$

Hence $(x + 1)(y + 1) = 1$. Consequently x is a semiunit.

Theorem 3.8

In the group ring Z_2G, where G is a cyclic group of order $2n$ (n is odd) generated by g, the element $x = g + g^2 + g^3 + \cdots + g^{2n-2}$ is a semiunit.

Proof. Take $y = g^2 + g^3 + g^4 + \cdots + g^{2n-1}$, We describe the multiplication $(x+1)(y+1)$ in the following array say A:

$$A = \begin{pmatrix} 1 & \boxed{g} & g^2 & g^3 & \cdots & g^{2n-5} & g^{2n-4} & g^{2n-3} & g^{2n-2} \\ g^2 & g^3 & g^4 & g^5 & \cdots & g^{2n-3} & g^{2n-2} & \boxed{g^{2n-1}} & 1 \\ \boxed{g^3} & g^4 & g^5 & g^6 & \cdots & g^{2n-2} & g^{2n-1} & 1 & g \\ g^4 & g^5 & g^6 & g^7 & \cdots & g^{2n-1} & 1 & g & \boxed{g^2} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ g^{2n-4} & g^{2n-3} & g^{2n-2} & g^{2n-1} & \cdots & g^{2n-9} & g^{2n-8} & g^{2n-7} & g^{2n-6} \\ \boxed{g^{2n-3}} & g^{2n-2} & g^{2n-1} & 1 & \cdots & g^{2n-8} & g^{2n-7} & g^{2n-6} & g^{2n-5} \\ g^{2n-2} & g^{2n-1} & 1 & g & \cdots & g^{2n-7} & g^{2n-6} & g^{2n-5} & \boxed{g^{2n-4}} \\ g^{2n-1} & 1 & g & g^2 & \cdots & g^{2n-6} & g^{2n-5} & g^{2n-4} & g^{2n-3} \end{pmatrix}$$

Hence $(x+1)(y+1) = 1$ and x is a semiunit.

Now, we introduce the concept of Strongly semiunit element in rings.

Definition 3.9

Let R be a commutative ring with identity. A semiunit x in R is said to be strongly semiunit if there exist semiunits a and b in R such that $ax = y$ and $by = x$.

In what follows we prove that in Z_p and Z_{2p}, p is an odd prime, every semiunit is strongly semiunit.

Theorem 3.10

In Z_p , p is an odd prime, every semiunit is a strongly semiunit.

Proof. Let $x \in Z_p$ be a semiunit. By Proposition 2.1, $x \neq p-1$. If $x = p-2$, take $y = p-2$ and $a = b = 1$, then

$$(x+1)(y+1) = 1,$$

therefore $ax \equiv y \ (mod\ p)$ and

$$by \equiv x \ (mod\ p),$$

hence x is a Strongly semiunit.

If $0 < x < p-2$, then $1 < x+1 < p-1$, hence by Proposition 2.1, x is a semiunit, then there exists $y \in Z_p$ such that $(x+1)(y+1) \equiv 1 (mod\ p)$. Now, consider the linear congruences

$$\beta y \equiv x\ (mod\ p) \qquad \ldots (1) \text{ and}$$

$$\alpha x \equiv y \ (mod \ p) \quad \ldots (2).$$

Since $gcd(y,p) = 1$, by Theorem 1.8, the linear congruence (1) has a unique solution, say a. We will show that a is a semiunit. If $a = p - 1$, then $(p-1)\,x \equiv y \ (mod\ p) \ldots(3)$. But $(x+1)(y+1) \equiv 1(mod\ p)$, hence

$$(p - y + 1)(y + 1) \equiv 1 \ (mod \ p).$$

Then

$$py + p - y^2 - y + y + 1 \equiv 1 \ (mod \ p),$$

thus $$1 - y^2 \equiv 1 \ (mod \ p),$$

implies that $$y \equiv 0 \ (mod \ p),$$

so $$x \equiv y \equiv 0 \ (mod\ p)\ ,$$

which is a contradiction with $0 < x < p - 2$, hence $a \neq p - 1$, so a is a semiunit element.

Similarly for the linear congruence (2). Hence x is a Strongly semiunit.

Finally if $x = 0$, take $y = 0$, then for $a = b = 1$ we get

$(x+1)(y+1) \equiv 1 \ (mod\ p)$, $ax \equiv y \ (mod\ p)$ and $by \equiv x \ (mod\ p)$ which means $x = 0$ is a Strongly semiunit.

Theorem 3.11

In Z_{2p} , p is an odd prime, every semiunit is a Strongly semiunit.

Proof. p has the form $4n + 1$ or it has the form $4n + 3$. We prove the case where p has the form $4n + 3$. Let $0 \neq x = 2k$ be a semiunit. Then either $1 \leq k \leq \frac{p-3}{2}$ or $\frac{p+1}{2} \leq k < p$. In case $1 \leq k \leq \frac{p-3}{2}$, put

$a = p - 2k - 1$ and consider the linear congruence

$$(p - 2k - 1)y \equiv 2k \ (mod \ 2p) \ \ldots (1),$$

Since $gcd(a, 2p) = 2$, by Theorem 1.8, this congruence has exactly two incongruent solutions. Suppose y_0 is a solution for the congruence (1). If y_0 is even, then it is a semiunit.

From (1) we have

$(p - 2k - 1)\, y_0 \equiv 2k \ (mod\ 2p)$. This means that

$$2p \,|\, 2k - py_0 + 2ky_0 + y_0 \ \ldots (2)$$

Now, $$(2k+1)(y_0 + 1) = 2ky_0 + y_0 + 2k + 1$$

$$\equiv 2ky_0 + 2k + y_0 - py_0 + 1 (mod\ 2p)$$

$\equiv 1(mod\ 2p)$, by (2) and y_0 is even.

Hence $(x+1)\,(y_0+1) \equiv 1\ (mod\ \ 2p)$.

If y_0 is odd, then y_0 is not a semiunit. Take $y_1 = y_0 + p$, then y_1 is a semiunit and $ay_1 \equiv 2k\ (mod\ \ 2p)$, similarly we get that

$$(x+1)(y_1+1) \equiv 1\ (mod\ 2p)\ .$$

Now, Consider the linear congruence

$$bx \equiv y_0\ (mod\ \ 2p)\ \ \ldots (3),$$

then $2bk \equiv y_0\ (mod\ \ 2p)$. Since $gcd(2k, 2p) = 2$ and $2 \mid y_0$, by Theorem 1.8, the linear congruence (3) has exactly two incongruent solutions. Suppose b_0 is a solution, similar to y_0 is a solution of (1), if b_0 is even then b_0 is semiunit, then the proof is complete. If b_0 is odd, take $b_1 = b_0 + p$, which is a semiunit and it is a solution of (3). Similarly one can prove the case when $\frac{p+1}{2} \leq k < p$.

Theorem 3.12

In Z_{p^2}, p is an odd prime, an element $0 \neq x$ is a strongly semiunit if it is of the form $kp+r$, $0 \leq k \leq p-1$ and $1 \leq r \leq p-2$.

Proof. Let $x = kp + r$, by Proposition 2.1, x is a semiunit.

Let $\beta = p^2 - kp - (r+1)$. Since $\beta + 1 = p^2 - kp - r$ is a unit, then β is a semiunit.

Now, consider the linear congruence

$$\beta y \equiv x\ \ (mod\ \ p^2)$$

$$\beta y \equiv kp + r\ \ (mod\ \ p^2)\quad \cdots (1)\ .$$

Since $gcd(\beta, p^2) = 1$, by Theorem 1.8, the congruence (1) has a unique solution say y_0.

Then $\beta y_0 \equiv x\ (mod\ p^2)$.

so $kp + r - (p^2 - kp - r - 1)y_0 \equiv 0\ \ (mod\ p^2)\ \cdots (2)$

We claim that y_0 is a semiunit and

$$(x+1)(y_0+1) \equiv 1\ (mod\ \ p^2)\ \ \cdots (3),$$

Now,

$$(x+1)(\,y_0+1) - 1 = (kp + r + 1)(\,y_0+1) - 1$$

$$= kpy_0 + ry_0 + y_0 + kp + r + 1 - 1$$

$$\equiv kp + r - (p^2 - kp - r - 1)y_0\ (mod\ p^2)$$

$$\equiv\ 0\ (mod\ \ p^2)\quad \text{by (2)}$$

Hence $(x+1)(y_0+1) \equiv 1\ (mod\ \ p^2)$ and y_0 is a semiunit.

Consider the linear congruence

$$\alpha\, x \equiv y_0\ (mod\ p^2)\ \cdots\ (4),$$

equivalently $\alpha\ (kp+r)\ \equiv\ y_0\ (mod\ \ p^2)$. Since $gcd(x, p^2) = 1$, by Theorem 1.8, The linear congruence (4) has a unique solution, say α_0 .

Hence $$\alpha_0 x \equiv y_0 (mod\ p^2)\ \cdots (5).$$

We claim that α_0 is a semiunit.

Suppose $\alpha_0 + 1$ is not a unit, so $p \mid \alpha_0 + 1$, hence

$$\alpha_0 + 1 \equiv 0\ (mod\ p)$$

$$(\alpha_0 + 1)\ x \equiv 0\ (mod\ p)$$

$$\alpha_0 x + x \equiv 0\ \ (mod\ \ p)$$

From linear congruence (5), we get $y_0 + x \equiv 0\ (mod\ \ p)$ ⋯(6)

From (3) we have

$$xy_0 + x + y_0 \equiv 0\ (mod\ \ p^2)$$

From (6) it follows that

$$xy_0 \equiv 0\ (mod\ \ p)$$

So $x^2 \equiv 0(mod\ p)$, then $x \equiv 0(mod\ p)$ which is a contradiction.

Thus $\alpha_0 + 1$ is not divisible by p, so α_0 is a semiunit, hence x is strongly semiunit.

4. Tri-regular Elements

In this section we study tri-regular elements in rings and we discuss its relation with some other elements , such as idempotent, Smarandache unit and non-trivial tripotent elements. Also we give a condition under which a unit is a tri-regular element.

Definition 4.1

Let R be a commutative ring with identity. An element x in R is a tri-regular element if there exists $y \in R \setminus \{0,1\}$ such that $x = x^3 y$. The element y is said to be a co-tri-regular element.

Remark 4.2

1. In any ring R, if x is an idempotent, then $1-x$ and x is tri-regular elements. For the idempotent x take $y = x$.
2. If α is a tripotent element of a ring R and $\alpha^2 \neq 1$, by Lemma 1.14**,** α^2 is a non-trivial

Smarandache idempotent, and by Remark 1.5, we have $1-\alpha^2$ is an idempotent, by (1), α^2 and $1-\alpha^2$ are tri-regular elements.

3. If y and $2y+1$ are nontrivial tripotents of Z_n and $gcd(n,12)=1$, by Proposition 1.6, $y+1$ is a nontrivial idempotent and $y \neq \frac{n}{2}$ for even n, hence by (1), $y+1$ is a tri-regular element.
4. If α is a supper idempotent by Definition 1.4, $\alpha^2-\alpha$ is a non-trivial idempotent, hence by Remark 1.5, we have $1-(\alpha^2-\alpha)$ is an idempotent, by (1), $\alpha^2-\alpha$ and $1-(\alpha^2-\alpha)$ are tri-regular elements.

Proposition 4.3

If R is an integral domain and $0 \neq x$ is a tri-regular element, then the co-tri-regular element y of x is the inverse of x^2.

Proof. Since $x^3y=x$, then $x(x^2y-1)=0$, but $x \neq 0$ hence $x^2y-1=0$ implies $x^2y=1$, which means that y is the inverse of x^2.

Proposition 4.4

If x is a unit in a commutative ring R with identity 1, then x is a tri-regular element if and only if $x^2 \neq 1$.

Proof: Suppose x is a unit and $x^2 \neq 1$, then there exists $y \neq x$ such that $xy=1$, hence $x^3y^2=x$. Hence x is a tri-regular element.

Now, suppose x is a tri-regular element and $x^2=1$.

Then there exists $y \in R \setminus \{0,1\}$ such that $x^3y=x$, then $x(y-1)=0$, but x is a unit, hence $y-1=0$ implies $y=1$, which is a contradiction. Hence a unit x in R is a tri-regular element if and only if $x^2 \neq 1$.

Corollary 4.5

Let R be a commutative ring with identity 1. Then every Smarandache unit is a tri-regular element.

Proof. Suppose x is a Smarandache unit, then x is a unit and by Theorem1.10, $x^2 \neq 1$, thus by Proposition 3.4, x is a tri-regular element.

Remark 4.6

Let F be a prime field of characteristic zero. By Theorem 1.11, every unit in F is a Smarandache unit, hence by Corollary 3.5, every unit in F is a tri-regular element.

Remark 4.7

In a ring R with identity 1, if $x \neq 1$ and $x^3=1$, then x is a tri-regular element, since for $y=x$ we have $x^3y=x$.

Proposition 4.8

Let R be a p-ring with identity, then a non-zero element $1 \neq x$ in R is a tri-regular element, if $x^{p-3} \neq 1$.

Proof. Suppose R is a p-ring and $x \in R$, by Definition 1.19, $x^p = x$ and $x^{p-3} \neq 1$. Then $x^3 x^{p-3} = x$, since $x^{p-3} \neq 1$, then x is a tri-regular element.

Example 4.9

1. Let R be a Boolean ring (Definition1.7). Then every element $x \in R$ is an idempotent, and by Remark 3.2(1), every element of R is a tri-regular element.
2. If R is an E-ring of degree 2, then also every element in R is a tri-regular element.

Proposition 4.10

Let R_1 and R_2 be two commutative rings with identity, and ϕ be a one to one homomorphism from R_1 into R_2 and $\phi(1_{R_1}) = 1_{R_2}$. If x is a tri-regular element of R_1, then $\phi(x)$ is a tri-regular element of R_2.

Proof. Since x is a tri-regular element, then there exists $y \in R_1 \backslash \{0_{R_1}, 1_{R_1}\}$ such that $x^3 y = x$. Since ϕ is a one to one, hence $\phi(y) \in R_2 \backslash \{0_{R_2}, 1_{R_2}\}$.

Now, $$\left(\phi(x)\right)^3 \phi(y) = \phi(x^3 y)$$
$$= \phi(x).$$

Hence $\phi(x)$ is a tri-regular element.

Proposition 4.11

Let R_1 and R_2 be two commutative rings with identity. If $x \in R_1$ and $y \in R_2$ are tri-regular elements, then $(x, y) \in R_1 \times R_2$ is a tri-regular element.

Proof. Since $x \in R_1$ is a tri-regular element, then there exists $a \in R_1 \backslash \{0,1\}$ such that $x^3 a = x$, similarly there exists $b \in R_2 \backslash \{0,1\}$ such that $y^3 b = y$. Then $(x, y)^3 (a, b) = (x^3 a, y^3 b)$

$$= (x, y) \in R_1 \times R_2.$$

Now, we study tri-regular elements in Z_n.

5. Tri-reguler Elements in $Z_n$and in group ring $Z_2 G$

In this section we study tri-reguler elements in Z_n and in $Z_2 G$, where G is a finite group with even order. We give necessary and sufficient condition under which an element of Z_n is tri-regular.

Proposition 5.1

In Z_p, p is a prime, an element x is a tri-regular element if and only if $1 < x < p - 1$.

Proof: Let $1 < x < p-1$ and consider the linear congruence

$$x^3 y \equiv x \ (mod\ p) \cdots (1).$$

Clearly $gcd(x,p) = 1$, then

$$x^2 y \equiv 1 \ (mod\ p) \cdots (2).$$

Since $gcd(x^2,p) = 1$, by Theorem 1.8, the congruence (2) has a unique solution which is a solution of (1). Hence x is a tri-regular element.

If $x = p-1$, then since $x^2 = 1$, by Proposition 3.4, x is not a tri-regular element and clearly $x = 1$ is not a tri-regular element.

Proposition 5.2

In Z_{p^2}, p is a prime, an element x is a tri-regular element if and only if x is not a zero divisor and $x \neq p^2 - 1$.

Proof: If x is a unit, then $gcd(x,p^2) = 1$. Consider the congruence

$$x^3 y \equiv x \ (mod\ p^2) \cdots (1), \text{ then}$$

$$x^2 y \equiv 1 \ (mod\ p^2) \cdots (2),$$

but $gcd(x^2,p^2) = 1$. by Theorem 1.8, the congruence (2) has a unique solution which is a solution of the congruence (1) and x is a tri-regular element.

If x is a zero divisor then x is of the form lp for $1 \leq l \leq p-1$. Then $x^3 \equiv 0 \ (mod\ p^2)$, hence there is no $y \in Z_{p^2} \setminus \{0,1\}$ such that $x^3 y \equiv x \ (mod\ p^2)$ which implies that x is not a tri-regular element.

Proposition 5.3

In Z_n, with $n = p_1 p_2 \cdots p_r$ and $p_1, p_2, \cdots, p_r$ are distinct primes, an element x is a tri-regular element if and only if $x^2 \neq 1$.

Proof: First suppose x is a unit. By Proposition 3.4, x is a tri-regular element if and only if $x^2 \neq 1$.

Now, suppose x is not a unit and $p_1 p_2 \cdots p_k \mid x$ and $p_j \nmid x$ for $j > k$, then $x = p_1 p_2 \cdots p_k l$, such that $p_j \nmid l$ for $k+1 \leq j \leq$ r.

Consider the congruence

$$x^3 y \equiv x \ (mod\ n) \cdots (1),$$

Then

$$(p_1 p_2 \cdots p_k l)^2 y \equiv 1 \ (mod\ p_{k+1} \cdots p_r) \cdots (2) .$$

Since $gcd((p_1 \cdots p_r l)^2 \, , \ p_{k+1} \cdots p_r) = 1$, by Theorem 1.8, the congruence (2) has a unique solution which is a solution of the congruence (1).

Hence x is a tri-regular element.

Theorem 5.4

In Z_n with the prime factorization of $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_r}^{\alpha_r}$, $\alpha_i > 1$ for some i, a zero divisor x is a tri-regular element if and only if $p_i \mid x$, implies ${p_i}^{\alpha_i} \mid x$ for each $i = 1, \dots, r$.

Proof: Suppose $p_i \mid x$ implies ${p_i}^{\alpha_i} \mid x$ without loss of generality. Then there exists $k < r$

such that

$x = l{p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$, with $p_j \nmid l$ for $k+1 \le j \le r$. Consider the congruence

$$x^3 y \equiv x\ (mod\ n) \qquad \cdots (1),$$

then

$$l^2{p_1}^{2\alpha_1}{p_2}^{2\alpha_2}\cdots{p_k}^{2\alpha_k}y \equiv 1\ (mod\ {p_{k+1}}^{\alpha_{k+1}}\cdots{p_r}^{\alpha_r}) \cdots (2),$$

but $gcd\left(l^2{p_1}^{2\alpha_1}{p_2}^{2\alpha_2}\cdots{p_k}^{2\alpha_k}, {p_{k+1}}^{\alpha_{k+1}}\cdots{p_r}^{\alpha_r}\right) = 1$, hence by Theorem 1.8, the congruence (2) has a unique solution which is a solution of the congruence (1), therefore x is a tri-regular element.

Now, suppose x is a tri-regular element and ${p_i}^t \mid x$ for some but ${p_i}^{\alpha_i} \nmid x$. Then $x = {p_i}^t l$ with $p_i \nmid l$.

Since x is a tri-regular element, then there exists $y \in R\backslash\{0,1\}$, such that

$$x^3 y \equiv x(mod\ n), \text{ then}$$

$$({p_i}^t l)^3 y \equiv {p_i}^t l\ (mod\ {p_1}^{\alpha_1}\cdots{p_i}^{\alpha_i}\cdots{p_r}^{\alpha_r}) \quad \cdots (1)\ .$$

If $gcd(l, n) = 1$, then

$$({p_i}^t l)^2 y \equiv 1(mod\ {p_1}^{\alpha_1}\cdots{p_i}^{\alpha_i - t}\cdots{p_r}^{\alpha_r}) \quad \cdots (2)$$

but $gcd(({p_i}^t l)^2, {p_1}^{\alpha_1}\cdots{p_i}^{\alpha_i - t}\cdots{p_r}^{\alpha_r}) = {p_i}^d$ for $d \le 2t$ and $p_i \nmid 1$, by Theorem 1.8, the congruence(2) has no solution which is a contradiction with x is a tri-regular element.

If $gcd(l, n) \neq 1$, then l is divisible by p_j, for some j.

Let $l = {p_1}^{s_1}\cdots{p_{i-1}}^{s_{i-1}}{p_{i+1}}^{s_{i+1}}\cdots{p_r}^{s_r}$, at least one of $s_m > 0$. Then

$$({p_i}^t{p_1}^{s_1}\cdots{p_{i-1}}^{s_{i-1}}{p_{i+1}}^{s_{i+1}}\cdots{p_r}^{s_r})^3 y \equiv {p_i}^t{p_1}^{s_1}\cdots{p_{i-1}}^{s_{i-1}}{p_{i+1}}^{s_{i+1}}\cdots{p_r}^{s_r}\ (mod\ {p_1}^{\alpha_1}\cdots{p_i}^{\alpha_i}\cdots{p_r}^{\alpha_r}),$$

$$({p_i}^t{p_1}^{s_1}\cdots{p_{i-1}}^{s_{i-1}}{p_{i+1}}^{s_{i+1}}\cdots{p_r}^{s_r})^2 y \equiv 1(mod\ {p_1}^{\alpha_1 - s_1}\cdots{p_{i-1}}^{\alpha_{i-1} - s_{i-1}}{p_i}^{\alpha_i - t}{p_{i+1}}^{\alpha_{i+1} - s_{i+1}}\cdots{p_r}^{\alpha_r - s_r}),$$

Then

$gcd(({p_i}^t{p_1}^{s_1}\cdots{p_{i-1}}^{s_{i-1}}{p_{i+1}}^{s_{i+1}}\cdots{p_r}^{s_r})^2, {p_1}^{\alpha_1 - s_1}\cdots{p_{i-1}}^{\alpha_{i-1} - s_{i-1}}{p_i}^{\alpha_i - t}{p_{i+1}}^{\alpha_{i+1} - s_{i+1}}\cdots{p_r}^{\alpha_r - s_r}) = ({p_i}^t{p_1}^{s_1}\cdots{p_{i-1}}^{s_{i-1}}{p_{i+1}}^{s_{i+1}}\cdots{p_r}^{s_r})^f$ for $f = 1,2$, but $({p_i}^t{p_1}^{s_1}\cdots{p_{i-1}}^{s_{i-1}}{p_{i+1}}^{s_{i+1}}\cdots{p_r}^{s_r})^f \nmid 1$, by Theorem 1.8, has no solution which is a contradiction with x is a tri-regular element.

Proposition 5.5

n Z_2G, G is a cyclic group of order n generated by g, then g^i for $i = 1, 2, \cdots, n-1$ is a tri-regular element.

Proof. Suppose $x = g^i$ and take $y = g^{n-2i}$, then $x^3y = x$. Hence x is a tri-regular element.

Remark 5.6

1. If x is a Smarandache idempotent of the group ring Z_2G, where G is a cyclic group of order n generated by g, by Remark 3.2(1), x is a tri-regular element, and by Proposition 1.14, $1 + x$ is also a Smarandache idempotent consequently by Remark 3.2(1), $1 + x$ is a tri-regular element.
2. Let Z_2G be the group ring, where $G = \langle g, g^{2p} = 1\rangle$ is a cyclic group of order $2p$ generated by g, p is a Mersenne prime. By Theorem 1.15, the group ring Z_2G has tri-regular elements and every element of the form $\alpha = g^{2l} + g^{2^2l} + \cdots + g^{2^{k-1}l} + g^{2^kl}$, is a Smarandache idempotent, hence by Remark 3.2(1), every element of this form is a tri-regular element.
3. Let Z_2G be the group ring of a finite cyclic group G of order 2^np, p is a prime of the form $2^nt + 1$ generated by g, by Theorem1.16, Z_2G has exactly two nontrivial Smarandache idempotents and $x = 1 + g^{2^n} + g^{2.2^n} + \cdots + g^{(p-1)2^n}$ is a non-trivial Smarandache idempotent hence it has at least two tri-regular elements, and by Proposition1.13, $x + 1$ is also a Smarandache idempotent, hence by Remark 3.2(1), x and $1 + x$ are tri-regular elements.
4. The group ring Z_nG, n is odd number and G is a cyclic group of order $n + 1$ generated by g, by Theorem 1.17, Z_2G has at least two non-trivial Smarandache idempotents, where $\alpha_1 = \sum_{i=0}^{n} g^i$, $\alpha_2 = \sum_{i=0}^{\frac{n-1}{2}}(g^{2i} + (n-1)g^{2i+1})$ and $\alpha_3 = \alpha_1 + \alpha_2 = \sum_{i=0}^{\frac{n-1}{2}} 2g^i$ are Smarandache idempotents of Z_nG, hence it has at least three tri-regular elements, so by Remark 3.2(1), α_1, α_2 and α_3 are tri-regular elements.
5. Let F be a field of characteristic zero and G any group of finite order generated by g, by Theorem 1.18, the group ring FG has a Smarandache idempotent, by Remark 3.2(1), it has a tri-regular element.

Proposition 5.7

In Z_2G, G is a cyclic group of order $2n$ (n is odd) generated by g, the two elements $x = 1 + g^2 + g^4 + \cdots + g^{2n-4} + g^{2n-2}$ and $y = g + g^3 + g^5 + \cdots + g^{2n-3} + g^{2n-1}$ are tri-regular elements.

Proof. Clearly x is an idempotent, hence by Remark 3.2(1), x is a tri-regular element.

$$y^2 = g^2 + g^6 + g^{10} + \cdots + g^{2n-8} + g^{2n-4} + 1 + g^2 + g^4 + g^{2n-6} + g^{2n-2}$$

$$= x$$

To find y^3 we must find xy, so we describe the multiplication in the following array $A =$

$$\begin{pmatrix}
\boxed{g} & g^3 & g^5 & g^7 & \cdots & g^{2n-7} & g^{2n-5} & g^{2n-3} & g^{2n-1} \\
g^3 & g^5 & g^7 & g^9 & \cdots & g^{2n-5} & g^{2n-3} & g^{2n-1} & g \\
\boxed{g^5} & g^7 & g^9 & g^{11} & \cdots & g^{2n-3} & g^{2n-1} & g & g^3 \\
g^7 & g^9 & g^{11} & g^{13} & \cdots & g^{2n-1} & g & g^3 & g^5 \\
\boxed{g^9} & g^{11} & g^{13} & g^{15} & \cdots & g & g^3 & g^5 & g^7 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
g^{2n-7} & g^{2n-5} & g^{2n-3} & g^{2n-1} & \cdots & g^{2n-15} & g^{2n-13} & g^{2n-11} & g^{2n-9} \\
\boxed{g^{2n-5}} & g^{2n-3} & g^{2n-1} & g & \cdots & g^{2n-13} & g^{2n-11} & g^{2n-9} & g^{2n-7} \\
g^{2n-3} & g^{2n-9} & g^{2n-7} & g^{2n-9} & \cdots & g^{2n-11} & g^{2n-9} & g^{2n-7} & g^{2n-5} \\
\boxed{g^{2n-1}} & g & \boxed{g^3} & g^5 & \cdots & g^{2n-9} & \boxed{g^{2n-7}} & g^{2n-5} & \boxed{g^{2n-3}}
\end{pmatrix}$$

Hence $y^3 = xy = y$, and we get $y^3x = y$ and y is a tri-regular element.

Proposition 5.8

In Z_2G, G is a cyclic group of order $2n$ (n is odd integer greater than 1) generated by g, if x is a tri-regular element and $x^3 = g + g^3 + g^5 + \cdots + g^{2n-3} + g^{2n-1}$, then either $x = g + g^3 + g^5 + \cdots + g^{2n-3} + g^{2n-1}$ or $x = 1 + g^2 + g^4 + \cdots + g^{2n-4} + g^{2n-2}$.

Proof. Suppose $x = a_0 + a_1g + a_2g^2 + \cdots + a_{2n-1}g^{2n-1}$ is a tri-regular element and

$$x^3 = g + g^3 + g^5 + \cdots + g^{2n-3} + g^{2n-1}.$$

Then there exists $y \in Z_2G\backslash\{0,1\}$,

$$y = b_0 + b_1g + b_2g^2 + \cdots + b_{2n-1}g^{2n-1},$$

such that $x^3y = x$. Now,

$$\begin{aligned}
x^3y = {} & (b_1 + b_3 + b_5 + \cdots + b_{2n-3} + b_{2n-1}) + \\
& (b_0 + b_2 + b_4 + \cdots + b_{2n-4} + b_{2n-2})g + \\
& (b_1 + b_3 + b_5 + \cdots + b_{2n-3} + b_{2n-1})g^2 + \cdots + \\
& (b_1 + b_3 + b_5 + \cdots + b_{2n-3} + b_{2n-1})g^{2n-2} + \\
& (b_0 + b_2 + b_4 + \cdots + b_{2n-4} + b_{2n-2})g^{2n-1} \\
= {} & a_0 + a_1g + a_2g^2 + \cdots + a_{2n-1}g^{2n-1}.
\end{aligned}$$

By equating the coefficients we get

$b_1 + b_3 + b_5 + \cdots + b_{2n-3} + b_{2n-1} = a_0$

$b_1 + b_3 + b_5 + \cdots + b_{2n-3} + b_{2n-1} = a_2$

$\vdots$

$b_1 + b_3 + b_5 + \cdots + b_{2n-3} + b_{2n-1} = a_{2n-4}$

$b_1 + b_3 + b_5 + \cdots + b_{2n-3} + b_{2n-1} = a_{2n-2}$ and

$b_0 + b_2 + b_4 + \cdots + b_{2n-4} + b_{2n-2} = a_1$

$b_0 + b_2 + b_4 + \cdots + b_{2n-4} + b_{2n-2} = a_3$

$\vdots$

$b_0 + b_2 + b_4 + \cdots + b_{2n-4} + b_{2n-2} = a_{2n-3}$

$b_0 + b_2 + b_4 + \cdots + b_{2n-4} + b_{2n-2} = a_{2n-1}$

Those two systems have solutions if and only if one of the following cases hold:

Case 1: $a_0 = a_2 = a_4 = \cdots = a_{2n-2} = 0$

and $a_1 = a_3 = a_5 = \cdots = a_{2n-1} = 1$,

$x = g + g^3 + g^5 + \cdots + g^{2n-3} + g^{2n-1}$.

Case 2: $a_0 = a_2 = a_4 = \cdots = a_{2n-2} = 1$

and $a_1 = a_3 = a_5 = \cdots = a_{2n-1} = 0$,

$x = 1 + g^2 + g^4 + \cdots + g^{2n-4} + g^{2n-2}$.

Case 3: $a_0 = a_2 = a_4 = \cdots = a_{2n-2} = 0$

and $a_1 = a_3 = a_5 = \cdots = a_{2n-1} = 0$, but this is a contradiction as x is a non-zero element.

Case 4: $a_0 = a_2 = a_4 = \cdots = a_{2n-2} = 1$

and $a_1 = a_3 = a_5 = \cdots = a_{2n-1} = 1$, but it is a contradiction since if

$x = \sum_{i=0}^{2n-1} g^i$, then $x^3 = 0$.

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