Shifted Genocchi Polynomials Operational Matrix for Solving Fractional Order System

Abdulnasir Isah[1] & Salisu Ibrahim[2]

[1&2] Department of Mathematics Education, Faculty of Education, Tishk International University, Erbil, Iraq
Correspondence: Abdulnasir Isah, Tishk International University, Erbil, Iraq.
Email: abdulnasir.isah@tiu.edu.iq

Doi: 10.23918/eajse.v7i1p74

Abstract: Genocchi polynomials are known to be defined on the interval [0, 1], but to benefit from the advantages of this polynomials in the field of fractional differential equations (FDEs), it was realized that fractional derivatives of many functions with arbitrary order cannot always be defined at x = 0. To avoid this difficulty, the idea of shifting from the interval [0, 1] to the interval [1, 2] makes it simple and applicable. Interestingly, almost all the properties of the Genocchi polynomials are inherited by the shifted polynomial. Therefore, in this research we construct shifted Genocchi polynomial's operational matrix of arbitrary order derivative and used it with the collocation method to solve some systems of FDEs.

Keywords: Shifted Genocchi Polynomials, Fractional Differential Equations, Operational Matrix, Collocation Method

1. Introduction

In recent years, fractional order calculus is being studied by many researchers, this is because many problems in engineering, physics, chemistry, biology, control theory and many more areas are now being formulated using FDEs. In this research we consider system of fractional differential equations (SFDEs) of the form.

$$D^{\alpha} y_j(x) = f\big(x, y_1(x), y_2(x), \dots y_n(x)\big) = g_j(x), \qquad y_j(0) = b_j \qquad [1]$$

Where b_j are constants and $j = 1,2, \dots, n$.

Even though, most of the SFDEs do not have exact analytic solutions, researchers worked broadly on various approximate and numerical techniques for their solutions. Decomposition method (Ray, Chaundhuri, & Bera 2006), Iteration method (Yang, Xiao, & Su 2010), Perturbation method (Odibat 2011), predictor-corrector method (Diethelm, Ford, & Freed 2002) are few of the earlier examples. It can be also noted that obtaining the solution of FDEs by some basic functions becomes widely used and some of the common of these basis functions include Block pulse functions, Bernulli polynomials, Bernstein polynomials, Laguerre polynomials, Legendre polynomials Genocchi polynomials, among others. In this article we used the so called shifted Genocchi polynomials which interestingly, inherits all the most important properties of the Genocchi polynomials (Isah & Phang 2019). We derived an

Received: April 10, 2021
Accepted: June 6, 2021
Isah, A., & Ibrahim, S. (2021). Shifted Genocchi Polynomials Operational Matrix for Solving Fractional Order System. *Eurasian Journal of Science & Engineering*, 7(1), 74-83.

operational matrix of arbitrary order derivative using the shifted Gennochi polynomial which we apply through collocation method to solve SFDEs (1). The results obtained are analogize with some standard results in literature to clearly clarify the veracity of our method.

The subsequent sections entail details as follows: Section two, introduce some useful definitions and properties of derivative and integration of fractional order we will used in the paper. Section three, Genocchi polynomials, shifted Genocchi polynomials and their properties are explained. Section four and five respectively, shows the derivation of the operational matrix of arbitrary order and its role for solving SFDEs through collocation points. In section six we solve an example and finally, conclusion is given in section seven.

2. Some Definitions and Properties

2.1 Fractional Derivatives and Integration

There are many definitions for fractional differentiation (Kilbas, Srivastava, & Trujillo 2006). The Riemann-Liouville definition has some dis-advantages when modelling some real-world problems (Podlubny, 1998). However, the Caputo's definition was introduced to cater such problems and so we make use of this definition which explained below.

Definition 2.1(Kilbas, Srivastava, & Trujillo 2006) The fractional derivative D^{α}in Caputo sense of a function f (x) is defined by:

$$D^{\alpha} f(x)=\frac{1}{\Gamma(n-\alpha)} \int_{0}^{x} \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}} d\tau, \ n-1<\alpha<n, \ n \epsilon \mathbb{N} \quad [2]$$

The Caputo fractional derivative has the following properties.

$$D^{\alpha} C=0, \qquad where\ C\ is\ constant \quad [3]$$

$$D^{\alpha} x^{\sigma}=\frac{\Gamma(\sigma+1)}{\Gamma(\sigma+1-\alpha)} t^{\sigma-\alpha}, if \quad \sigma \in \mathbb{N} \cup \{0\} \ and \ \sigma \geq \lceil\alpha\rceil$$

Where, $\lceil\alpha\rceil$ denote ceil function with respect to.

Caputo fractional operator is also linear since

$$D^{\alpha}\big(\lambda f(t)+\mu g(t)\big)= \lambda D^{\alpha} f(t)+\mu D^{\alpha} g(t). \quad [4]$$

with λ and μ constants.

3. Polynomials and Some Properties

Genocchi polynomials $G_n(x)$ is defined as in (Isah & Phang 2019), (Araci, 2012, 2014) and (Bayad & Kim 2010) by means of the generating function given by

$$Q(t, x)=\frac{2 t e^{x t}}{e^{t}+1}=\sum_{n=0}^{\infty} G_{n}(x) \frac{t^{n}}{n !}, \ \ (|t|<\pi) \quad [5]$$

where $G_n(x)$ is the Genocchi Polynomial of degree n obtained by

$$G_{n}(x)=\sum_{k=0}^{n}\binom{n}{k} G_{n-k} x^{k}. \quad [6]$$

G_{n-k} here is the Gennochi number which can also be obtained from

$$G_n = G_n(0), \tag{7}$$

Some of the properties of this polynomials are given as (Isah and Phang 2019).

$$\int_0^1 G_n(x) G_m(x) dx = \frac{2(-1)^n n!\, m!}{(m+n)!} G_{m+n}, \quad n, m \geq 1 \tag{8}$$

$$\frac{dG_n(x)}{dx} = nG_{n-1}(x), \quad n \geq 1 \tag{9}$$

$$G_n(1) + G_n(0) = 0, \quad n > 1 \tag{10}$$

3.1 Functions Approximation

Let f be a square integrable function I was already shown in (Isah & Phang, 2019) and (Isah &Phang, 2017) that there exist the unique coefficients $c_1, c_2, \cdots, c_N$ such that.

$$f \approx f^* = \sum_{n=1}^{N} c_n G_n(t) = \mathbf{CG}(t) \tag{11}$$

Were,

$$\mathbf{C} = [c_1, c_2, \cdots, c_N], \quad \mathbf{G}(t) = [G_1(t),\ G_2(t), \cdots, G_N(t)].$$

And the coefficients c_n can be obtained by as shown in (Isah & Phang, 2019) and (Isah & Phang, 2017)

$$c_n = \frac{1}{2n!}(f^{(n-1)}(1) + f^{(n-1)}(0)), \tag{12}$$

Where, $(n-1)$ in $f^{(n-1)}(x)$ denotes the $(n-1)^{th}$ derivative of f

However, equation (12) may fail when $f(x)$ involves a rational or irrational power of x. For instance, let $N = 3$, $f(x) = x^{3/2} \approx \sum_{n=1}^{3} c_n G_n(x) = c_1 G_1(x) + c_2 G_2(x) + c_3 G_3(x)$

$$\begin{aligned} c_3 &= \frac{1}{2(3!)}\left(\frac{d^2}{dx^2} x^{3/2}|_{x=0} + \frac{d^2}{dx^2} x^{3/2}|_{x=1}\right) \\ &= \frac{1}{2(3!)}\left(\frac{3}{4\sqrt{x}}|_{x=0} + \frac{3}{4\sqrt{x}}|_{x=1}\right) \end{aligned}$$

Which clearly shows that the coefficient is undefined.

To get rid of this kind of problems, we define the shifted Genocchi polynomials by shifting $G_n(x)$

from the interval [0,1] to the interval [1,2] by simply substituting $x = t - 1$ hence obtaining the analytical form as.

$$S_i(t) = \sum_{k=0}^{i}\sum_{r=0}^{k} \frac{(-1)^{k-r} i! G_{i-k}}{(i-k)!r!(k-r)!} t^r \quad [13]$$

The following are some of the properties inherited.

$$\frac{dS_n(x)}{dx} = \frac{n}{h} S_{n-1}(x), \quad n \geq 1 \quad [14]$$

$$S_n(1) + S_n(2) = 0, \quad n > 1 \quad [15]$$

Below we give the version of equation (12) for shifted Genocchi on the interval [1, 2].

Suppose that f is an arbitrary square integrable function on [1,2] approximated by the shifted Genocchi series $f(x) \approx \sum_{n=1}^{N} c_n S_n(x) = \mathbf{C}^T \mathbf{S}(x)$. Then, the c_n are calculated by.

$$c_n = \frac{1}{2(n!)}\left(f^{(n-1)}(1) + f^{(n-1)}(2)\right) \quad [16]$$

To see (16), suppose $f(x) = \sum_{k=1}^{N} c_k S_k(x)$. Using (14) and (15), one can inductively observed that.

$$f(1) + f(2) = c_1\big(S_1(1) + S_1(2)\big) + c_2\big(S_2(1) + S_2(2)\big) + \cdots + c_N\big(S_N(1) + S_N(2)\big) = 2c_1$$

Thus, $c_1 = \frac{1}{2}\left(f(1) + f(2)\right)$

$$f^{(1)}(1) + f^{(1)}(2) = c_2\big(S_1(1) + S_1(2)\big) + 3c_3\big(S_2(1) + S_2(2)\big) + \cdots + Nc_N\big(S_{N-1}(1) + S_{N-1}(2)\big) = 2c_2(2)$$

Thus, $c_2 = \frac{1}{2(2!)}\left(f^{(1)}(1) + f^{(1)}(2)\right)$

$$f^{(2)}(1) + f^{(2)}(2) = 3(2)c_3\big(S_1(1) + S_1(2)\big) + \cdots + N(n-1)c_N\big(S_{N-2}(1) + S_{N-2}(2)\big) = 3(2)c_3(2)$$

Thus, $c_3 = \frac{1}{2(3!)}\left(f^{(1)}(1) + f^{(1)}(2)\right)$

Repeating this inductively for $i = 1,2, \ldots N$ we have

$$c_i = \frac{i}{2(i!)}\left(f^{(i-1)}(1) + f^{(i-1)}(2)\right)$$

Before we derive the operational matrix, we observe the following important result.

Let $S_i(t)$ be the shifted Genocchi polynomial then, $D^\alpha S_i(t) = 0$ for $i = 1, \ldots, \lceil\alpha\rceil - 1,\ \alpha > 0$.

From Eq. (13) we have

$$S_i(t) = \sum_{k=0}^{i}\sum_{r=0}^{k} \frac{(-1)^{k-r} i!\, G_{i-k}}{(k-r)!\, r!\, (i-k)!} t^r$$

Now, for $i = 1, \ldots, \lceil\alpha\rceil - 1$ if $S_i(t) = C$, (C is any constant), then by Eq. (3) we obtain $D^\alpha S_i(t) = 0$.
Otherwise, by using Eq.(4) and Eq.(5) one can easily see that $D^\alpha S_i(t) = 0$,

for $i = 1, \ldots, \lceil\alpha\rceil - 1$, this completes the proof of the result.

4. Shifted Genocchi Operational Matrix of Fractional Derivatives

The theorem below shows how we derive the operational matrix for the shifted polynomials.

Theorem 1. If $\mathbf{S}(t)$ is a vector of shifted Genocchi then for $\alpha > 0$ we have

$$D^{\alpha}\mathbf{S}(t)^{T} = M^{\alpha}\mathbf{S}(t)^{T} \quad [17]$$

With,

$$\mathbf{M}^{(\alpha)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \rho_{\lceil\alpha\rceil,k,1} & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \rho_{\lceil\alpha\rceil,k,2} & \cdots & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \rho_{\lceil\alpha\rceil,k,N} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{i} \rho_{i,k,1} & \sum_{k=\lceil\alpha\rceil}^{i} \rho_{i,k,2} & \cdots & \sum_{k=\lceil\alpha\rceil}^{i} \rho_{i,k,N} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{N} \rho_{N,k,1} & \sum_{k=\lceil\alpha\rceil}^{N} \rho_{N,k,2} & \cdots & \sum_{k=\lceil\alpha\rceil}^{N} \rho_{N,k,N} \end{pmatrix} \quad [18]$$

where $\rho_{i,k,l}$ is given by

$$\rho_{i,k,l} = \sum_{r=0}^{k} \frac{(-1)^{k-r}(i)!\, G_{i-k}(f^{(l-1)}(1) + f^{(l-1)}(2))}{2l!\,(i-k)!\,(k-r)!\,\Gamma(r+1-\alpha)} \quad [19]$$

where G_{i-k} is the Genocchi number.

Proof:

Using Eqs. (4) and (13), we have

$$\begin{aligned} D^{\alpha}S_i(t) &= \sum_{k=1}^{i}\sum_{r=0}^{i-k} \frac{(-1)^{k-r} i!\, G_{i-k}}{(i-k)!\,(k-r)!\,r!} D^{\alpha}t^{r} \\ &= \sum_{k=\lceil\alpha\rceil}^{i}\sum_{r=0}^{i-k} \frac{(-1)^{i-k-r} i!\, G_{i-k}}{(i-k)!\,(k-r)!\,\Gamma(r+1-\alpha)} t^{r-\alpha}. \end{aligned} \quad [20]$$

Let $f(t) = t^{r-\alpha}$, now we can approximate $f(t)$ by truncated shifted series, as;

$$f(t) = \sum_{l=0}^{N} w_l\, S_l(t). \quad [21]$$

The constants w_l can be obtained as

$w_l = \frac{1}{2l!}(f^{(l-1)}(1) + f^{(l-1)}(2))$.

Therefore,
$f(t) = \sum_{l=0}^{N} \frac{1}{2l!}(f^{(l-1)}(1) + f^{(l-1)}(2))S_l(t)$, putting this in Eq. (20), we have.

$$\begin{aligned} D^{\alpha}S_i(t) = & \sum_{l=0}^{N}\left(\sum_{k=\lceil\alpha\rceil}^{i}\sum_{r=0}^{i-k}\frac{(-1)^{i-k-r} i!\, G_{i-k}(f^{(l-1)}(1) + f^{(l-1)}(2))}{2l!\,(i-k)!\,(k-r)!\,\Gamma(r+1-\alpha)}\right)S_l(t) \\ & = \sum_{l=0}^{N}\left(\sum_{k=\lceil\alpha\rceil}^{i}\rho_{i,k,l}\right)S_l(t) \end{aligned} \quad [22]$$

where $\rho_{i,k,l}$ is given in Eq. (19). Putting Eq. (22) in vector form, we get

$$D^{\alpha}S_i(t) = \left[\sum_{k=\lceil\alpha\rceil}^{i}\rho_{\lceil\alpha\rceil,k,1} \quad \sum_{k=\lceil\alpha\rceil}^{i}\rho_{\lceil\alpha\rceil,k,2} \quad \cdots \quad \sum_{k=\lceil\alpha\rceil}^{i}\rho_{\lceil\alpha\rceil,k,N}\right]S(t) \quad i = \lceil\alpha\rceil \cdots N. \quad [23]$$

Hence, we can write.

$$D^{\alpha}S_i(t) = [0,0,\cdots 0]S(t) \quad i = 1,\cdots,\lceil\alpha\rceil - 1. \quad [24]$$

Thus, combining Eq. (23) and Eq. (24) leads to the desired result.

5. Shifted Genocchi Operational Matrix and Collocation Points

Applying collocation method together with the shifted Genocchi operational matrix to numerically solve the SFDEs (1)

We first approximate $D^{\alpha}y_j(t),\ \ y_j(t)$ by means of shifted Genocchi polynomials as follows:

$$y_j(t) = \sum_{k=0}^{N} c_{j,k}\, S_k(t) = \mathbf{C}_j S(t)^T \quad [25]$$

where the vector $\mathbf{C}_j = [c_{j,1},\ \ c_{j,2},\cdots,c_{j,N}]$ is unknown. Now employing Eq. (17) on Eq. (25) we have

$$D^{\alpha}y_j(t) \simeq \mathbf{C}_j M^{(\alpha)}\mathbf{S}(t)^T, \quad j = 1,2,\cdots,n. \quad [26]$$

Therefore, substituting Eq. (25) and Eq. (26) in Eq. (1), we have

$$\mathbf{C}_j M^{(\alpha)}\mathbf{S}(t)^T = f_j(t,\ \ \mathbf{C}_1\mathbf{S}(t)^T\ \ \mathbf{C}_2\mathbf{S}(t)^T,\cdots,\mathbf{C}_n\mathbf{S}(t)^T) \quad j = 1,2,\cdots,n.. \quad [27]$$

For the initial conditions we first need to modify the initial point from 0 to 1, without lost of generality, we can let the initial condition now to be $y_j(1) = \gamma_j$. Thus, approximating our new initial condition with shifted Gennochi we have

$$\mathbf{C}_j\mathbf{S}(1)^T = \gamma_j \quad j = 1,2,\cdots,n. \quad [28]$$

For the solution of Eq. (1), we put the collocation points.

$t_i = \frac{i}{N-1} + 1, \quad i = 1,2,\cdots,N-1$ on Eq. (27) and we get

$$\mathbf{C}_j M^{(\alpha)} \mathbf{S}(t_i)^T = f_j(t_i, \ \mathbf{C}_1 \mathbf{S}(t_i)^T \ \mathbf{C}_2 \mathbf{S}(t_i)^T, \cdots, \mathbf{C}_n \mathbf{S}(t_i)^T) \quad [29]$$
$$i = 1,2,\cdots,N-1, \quad j = 1,2,\cdots,n.$$

Thus, Eq. (29) are $n(N-1)$ equations in $\mathbf{C}_i$. These equations and Eq. (28) make $n(N)$ equations which can be solved easily and consequently we can obtain $y_j(t)$ given in (Eq. (25).

6. Numerical Examples

Example 1. Let us consider the nonlinear SFDE.

$$D^\alpha y_1(t) = y_1(t) + \frac{y_2(t)}{2}$$
$$D^\alpha y_2(t) = \left(y_1(t)\right)^2 + y_2(t)$$

Subject to, $y_1(0) = 0, \quad y_2(0) = 0$

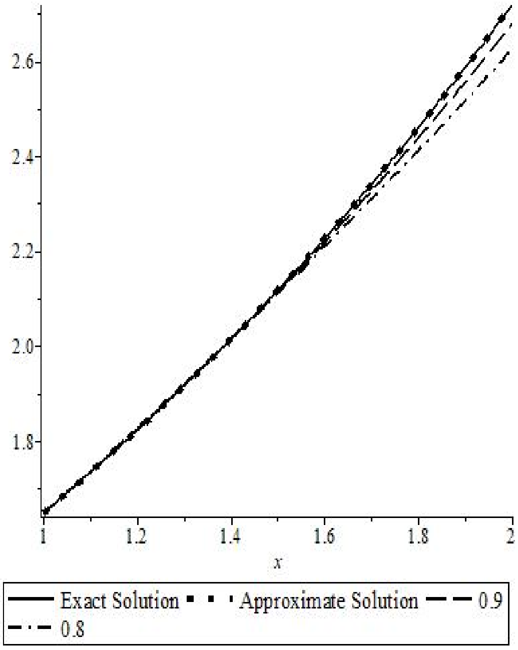
First, we reconsider the initial conditions on the interval [1, 2]. The new initial conditions now are $y_1(1) = 1.64872 \ and \ y_2(1) = 2.71828$

The system has exact solution when $\alpha = 1$ as $y_1(t) = e^{\frac{t}{2}}$ and $y_2(t) = te^t$. We solve this example when $N = 10$. The numerical solution and absolute errors for $y_1(t)$ and $y_2(t)$ are shown in Table 1.

This also solve this example for $\alpha = 0.9, and \ 0.8,$ the solutions are analogized with the exact solution when $\alpha = 1$ as shown in Figure 1. One can see that as α goes to 1, our solutions approach the exact solution.

Table 1: Numerical solution $y_1(t)$, and $y_2(t)$, in comparison with the exact solutions and absolute errors obtained

t	Exact $y_1(t)$	Exact $y_2(t)$	Appr.$y_1(t)$	Appr.$y_2(t)$	Abs. Error $y_1(t)$	Abs.Error$y_2(t)$
1.0	1.6487213	2.7182818	1.6487213	2.7182818	1.00000E-09	1.80000E-08
1.1	1.7332530	3.3045826	1.7332530	3.3045826	1.00000E-09	1.40000E-08
1.2	1.8221188	3.9841403	1.8221188	3.9841403	0.00000E+00	1.90000E-08
1.3	1.9155408	4.7700856	1.9155408	4.7700857	0.00000E+00	1.90000E-08
1.4	2.0137527	5.6772799	2.0137527	5.6772800	1.00000E-09	1.70000E-08
1.5	2.1170000	6.7225336	2.1170000	6.7225336	1.00000E-09	1.90000E-08
1.6	2.2255409	7.9248519	2.2255409	7.9248519	2.00000E-09	1.60000E-08
1.7	2.3396469	9.3057105	2.3396469	9.3057106	1.00000E-09	1.70000E-08
1.8	2.4596031	10.8893654	2.4596031	10.8893654	1.00000E-09	2.00000E-08
1.9	2.5857097	12.7031994	2.5857097	12.7031994	0.00000E+00	2.00000E-08
2.0	2.7182818	14.7781122	2.7182818	14.7781122	1.00000E-09	1.00000E-08



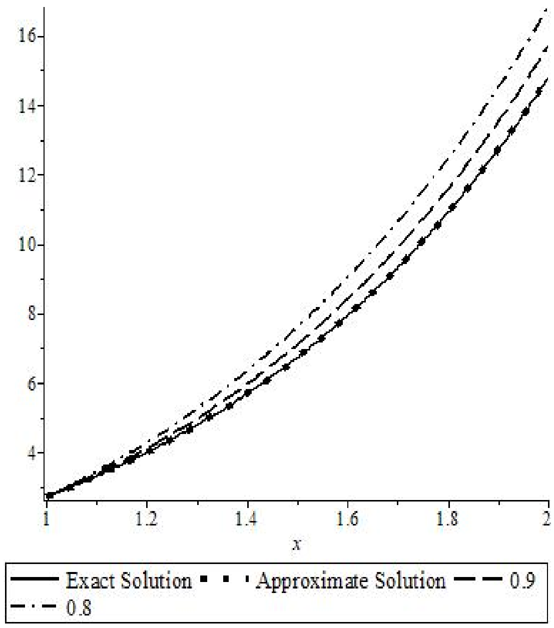


Figure 1: Numerical solutions $y_1(x)$ and $y_2(x)$ obtained respectively, when $\alpha = 0.8, 0.9\ and\ 1$ for example 1

It is clearly observed that when α goes closed to 1, the results go closer to the exact solution.

Example 2. Consider the following nonlinear SFDE.

$$D^{\alpha}y_1(t) = y_1(t)$$
$$D^{\alpha}y_2(t) = 2\big(y_1(t)\big)^2$$
$$D^{\alpha}y_3(t) = 3y_1(t)y_2(t)$$

Subject to, $y_1(0) = 1,\quad y_2(0) = 1,\quad y_3(0) = 0$

We reconsider the initial conditions on the interval [1, 2]. The new initial conditions now are $y_1(1) = 2.71828,\ y_2(1) = 7.38906\ and\ y_3(1) = 19.08554$. The exact solution of this system when $\alpha = 1$ is known to be $y_1(t) = e^t,\ y_2(t) = e^{2t}\ and\ y_3(t) = e^{3t} - 1$. This example is solved using our method with $N = 10$. The numerical results obtained and the absolute errors for $y_1(t),\ y_2(t)\ and\ y_3(t)$ are respectively shown in Table 2.

Table 2: Numerical solutions y1(t), y2(t) and y3(t) with absolute errors obtained for example 2

t	Appr. $y_1(t)$,	Abs Error$y_1(t)$	Appr.$y_2(t)$	Abs Error$y_2(t)$	Appr.$y_3(t)$	Abs Error$y_3(t)$
1.0	2.7182818	0.00000E+00	7.3890561	0.00000E+00	19.0855369	1.00000E-06
1.1	3.0041660	1.00000E-09	9.0250135	5.46000E-07	26.1126389	1.37830E-04
1.2	3.3201169	1.00000E-09	11.0231764	5.40000E-07	35.5982344	1.33560E-04
1.3	4.0552000	0.00000E+00	13.4637380	5.30000E-07	48.4024491	1.33330E-04
1.4	4.0552000	0.00000E+00	16.4446468	5.40000E-07	65.6863310	1.35910E-04
1.5	4.4816891	0.00000E+00	20.0855369	5.50000E-07	89.0171313	1.36560E-04
1.6	4.9530324	1.00000E-09	24.5325302	5.40000E-07	120.5104175	1.37700E-04
1.7	5.4739474	0.00000E+00	29.9641001	5.60000E-07	163.0219073	1.43500E-04
1.8	6.0496475	1.00000E-09	36.5982344	5.90000E-07	220.4064162	1.38900E-04
1.9	6.6858944	1.00000E-09	44.7011845	5.80000E-07	297.8674010	1.35000E-04
2.0	7.3890561	0.00000E+00	54.5981500	2.00000E-08	402.4287935	1.88000E-05

We also solved the example when $\alpha = 0.8, 0.9$ and the results are compared with the exact solution when $\alpha = 1$ as shown in figure 2 as shown below. It is clear from the figures that when α approaches 1, our results approach the exact solution.

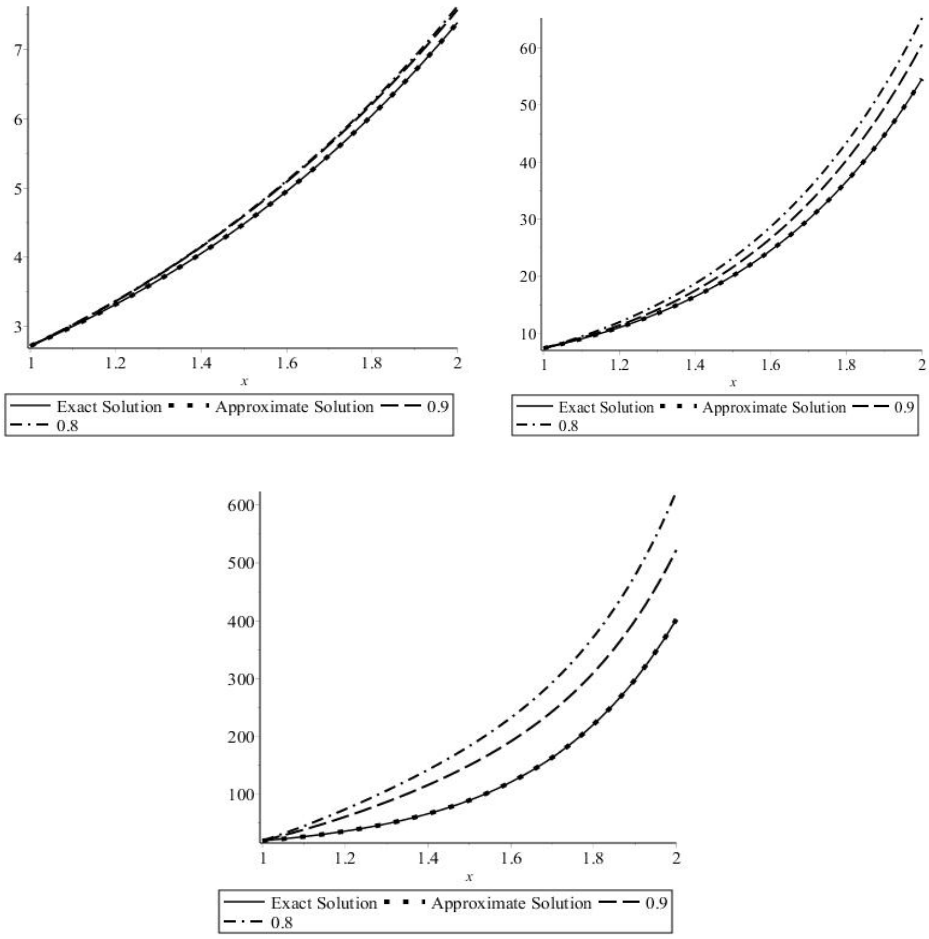


Figure 2: comparison of our solutions y1(x); y2(x) and y3(x) respectively, when $\alpha = 0.8, 0.9$ and 1 for example2

7. Conclusion

In this research, we obtained the operational matrix by the shifted Genocchi polynomials which is used through the collocation points to solve the SFDEs. The comparability of the results shows that the method is good and accurate procedure for obtaining the solutions of SFDEs.

References

Araci, S. (2012). Novel identities for *q*-Genocchi numbers and polynomials. *Journal of Function Spaces and Applications*, *2012*.

Araci, S. (2014). Novel identities involving Genocchi numbers and polynomials arising from applications of umbral calculus. *Applied Mathematics and Computation*, *233*, 599-607.

Bayad, A., & Kim, T. (2010). Identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials. *Adv. Stud. Contemp. Math*, *20*(2), 247-253.

Diethelm, K., Ford, N. J., & Freed, A. D. (2002). A predictor-corrector approach for the numerical solution of fractional differential equations. *Nonlinear Dynamics*, 29(1), 3-22.

Isah, A., & Phang, C. (2019). New operational matrix of derivative for solving non-linear fractional differential equations via Genocchi polynomials. *Journal of King Saud University-Science*, 31(1), 1-7.

Isah, A., & Phang, C. (2017, January). On Genocchi operational matrix of fractional integration for solving fractional differential equations. In *AIP Conference Proceedings* (Vol. 1795, No. 1, p. 020015). AIP Publishing LLC.

Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and applications of fractional differential equations* (Vol. 204). elsevier.

Odibat, Z. (2011). On Legendre polynomial approximation with the VIM or HAM for numerical treatment of nonlinear fractional differential equations. *Journal of Computational and Applied Mathematics*, 235(9), 2956-2968.

Podlubny, I. (1998). *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Elsevier.

Ray, S. S., Chaudhuri, K. S., & Bera, R. K. (2006). Analytical approximate solution of nonlinear dynamic system containing fractional derivative by modified decomposition method. *Applied mathematics and computation*, *182*(1), 544-552.

Yang, S., Xiao, A., & Su, H. (2010). Convergence of the variational iteration method for solving multi-order fractional differential equations. *Computers & Mathematics with Applications*, *60*(10), 2871-2879.