Semi Nilpotent Elements

Kurdistan M. Ali [1] & Parween A. Hummadi [2]

[1] Salahaddin University, College of Science, Department of Mathematics
[2] Salahaddin University, College of Education, Department of Mathematics

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Abstract: In this paper we study semi nilpotent elements in rings. It is shown that every element of $\mathbb{Z}_n$ where n is square free is a trivial semi nilpotent. It is proved that every nontrivial nilpotent element is a nontrivial semi nilpotent. Conditions are given under which every element of the group ring $\mathbb{Z}_nG$ is semi nilpotent. It is shown that if p is prime and p divides the order of G, then $\mathbb{Z}_pG$ has nontrivial semi nilpotent. Also it is proved that if G is a cyclic group of order q^n, then every element of $\mathbb{Z}_pG$, p is prime, is semi nilpotent.

Keywords: Nilpotent, Semi Nilpotent

1. Introduction

The concept of semi nilpotent element was introduced by Vasantha Kandasamy (1997). An element x of a ring $\mathcal{R}$ is semi nilpotent if $x^n - x$ is a nilpotent element of $\mathcal{R}$, for some positive integer $n > 1$. If $x^n - x = 0$, then x is said to be a trivial semi nilpotent. Clearly every nilpotent element is semi nilpotent and every idempotent is semi nilpotent. In Kandasamy (2002), it is shown that if K is a field of characteristic 0 and G is a torsion free abelian group, then the group ring KG has no nontrivial semi nilpotent element. In this work by using some well-known theorems in number theory the form of both trivial and nontrivial semi nilpotent elements in $\mathbb{Z}_n$ are given. It is shown that every element except 0 and 1 in the group ring $\mathbb{Z}_2G$ where G is a cyclic group of order 2^n, for even n, is a nontrivial semi nilpotent element. At the end we answer an open problem given in Kandasamy (2002) concerning such elements, **Theorem 2.10** and **Theorem 2.12**.

2. Semi Nilpotents

In this section, we study semi nilpotent elements in $\mathbb{Z}_n$, and in the group ring $\mathbb{Z}_nG$ for a cyclic group G of finite order.

Proposition 2.1. Every nontrivial nilpotent element in $\mathbb{Z}_n$, with the prime factorization of $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ is a nontrivial semi nilpotent.

Proof. Let $0 \neq x$ be a nilpotent element of $\mathbb{Z}_n$ and m be the least positive integer such that $x^m \equiv 0 \pmod{n}$. Then $x = p_1^{\ell_1} p_2^{\ell_2} \ldots p_k^{\ell_k}$, such that at least one of $\ell_i < \alpha_i$ say $\ell_1 < \alpha_1$. We have to show that $x^t - x \not\equiv 0 \pmod{n}$ for each positive integer t. Suppose $x^t - x \equiv 0 \pmod{n}$ for some

$t > 1$, so $x\,(x^{t-1} - 1) \equiv 0 \pmod{n}$, which means that $n \mid x\,(x^{t-1} - 1)$, which implies ${p_1}^{\alpha_1} \mid x\,(x^{t-1} - 1)$. But $\ell_1 < \alpha_1$, so ${p_1}^{\alpha_1} \nmid x$ and clearly $(x^{t-1} - 1)$ is not divisible by p_1, so ${p_1}^{\alpha_1} \nmid x\,(x^{t-1} - 1)$, contradiction.■

Proposition 2.2. Every element in $\mathbb{Z}_n$, where n is a square free is a trivial semi nilpotent element.

Proof. Since n is square free, then $n = p_1\, p_2 \dots p_k$ (Dummit & Foote, 2004), for some distinct primes p_i, $1 \le i \le k$. Now let $a \in \mathbb{Z}_n$. If $p_i \nmid a$, for each $1 \le i \le k$, then by Euler's Theorem (Burton, 1980), $a^{\varphi(p_1\, p_2 \dots p_k)} \equiv 1 \pmod{n}$, that is $a^{(p_1 - 1)(p_2 - 1)\dots(p_k - 1)} \equiv 1 \pmod{n}$, which means

$$a^{(p_1-1)(p_2-1)\dots(p_k-1)+1} \equiv a \pmod{n},$$

so a is a trivial semi nilpotent element in $\mathbb{Z}_n$. Now, suppose a is divisible by some of p_i's, without loss of generality, suppose a is divisible by $p_1, p_2, \dots, p_\ell$ and a is not divisible by $p_{\ell+1}, p_{\ell+2}, \dots p_k$. Hence $a = t_1\, p_1 = t_2\, p_2 = \cdots = t_\ell\, p_\ell$ for some $t_i \in \mathbb{Z}^+$ with $p_j \nmid t_i$, $1 \le j \le \ell$ and by Fermat's Little Theorem (Schroeder, 2006),

$$(t_1\, p_1)^{(p_{\ell+1}-1)\,(p_{\ell+2}-1)\dots(p_k-1)} \equiv 1 \pmod{p_{\ell+1} p_{\ell+2} \dots p_k},$$

$$p_{\ell+1} p_{\ell+2} \dots p_k \mid (t_1\, p_1)^{(p_{\ell+1}-1)(p_{\ell+2}-1)\dots(p_k-1)} - 1.$$

So,

$$t_1\, p_1\, p_{\ell+1} p_{\ell+2} \dots p_k \mid t_1\, p_1\, a^{(p_{\ell+1}-1)\,(p_{\ell+2}-1)\dots(p_k-1)} - t_1\, p_1.$$

But $p_i \mid t_1\, p_1$ for each $2 \le i \le \ell$, then

$$p_1\, p_2 \dots p_k \mid t_1\, p_1\, a^{(p_{\ell+1}-1)\,(p_{\ell+2}-1)\dots(p_k-1)} - t_1\, p_1,$$

this means $a^{(p_{\ell+1}-1)\,(p_{\ell+2}-1)\dots(p_k-1)+1} \equiv a \pmod{n}$.

Hence a is a trivial semi nilpotent element.■

Theorem 2.3. Consider $\mathbb{Z}_n$, with the prime factorization of $n = {p_1}^{t_1}\, {p_2}^{t_2} \dots {p_k}^{t_k}$, with at least one of $t_i > 1$ and let $0 \ne \omega \in \mathbb{Z}_n$. Then ω is a trivial semi nilpotent if ω has the form $\omega = {q_1}^{s_1}\, {q_2}^{s_2} \dots {q_\ell}^{s_\ell}\, {r_1}^{\alpha_1} {r_2}^{\alpha_2} \dots {r_t}^{\alpha_t}$, with $\ell < k$, $\alpha_i \ge 0$, and for each $1 \le i \le \ell$ there exists $1 \le j \le k$ such that $q_i = p_j$ and r_i are primes distinct from p_j's such that $s_i = 0$ for each i or $s_i \ge t_i$ for each i.

Proof. Without loss of generality, put $\omega = {p_1}^{s_1}\, {p_2}^{s_2} \dots {p_m}^{s_m} {r_1}^{\alpha_1} {r_2}^{\alpha_2} \dots {r_t}^{\alpha_t}$. If $s_i = 0$ for all $1 \le i \le m$, then $\gcd(\omega, n) = 1$, thus by Euler's Theorem,

$$(\omega)^{\varphi(n)} \equiv 1 \pmod{n}$$

$$(\omega)^{\varphi(n)+1} \equiv \omega \pmod{n}$$

Hence ω is a trivial semi nilpotent element.

If $s_i \ge t_i$ for $1 \le i \le m$, then by Euler's Theorem,

$$(\omega)^{\varphi({p_{m+1}}^{t_{m+1}}\, {p_{m+2}}^{t_{m+2}} \dots {p_k}^{t_k})} \equiv 1 \pmod{{p_{m+1}}^{t_{m+1}}\, {p_{m+2}}^{t_{m+2}} \dots {p_k}^{t_k}}$$

$$(\omega)^{\varphi({p_{m+1}}^{t_{m+1}}\,{p_{m+2}}^{t_{m+2}}\,\dots\,{p_k}^{t_k})+1} \equiv \omega \pmod{{p_{m+1}}^{t_{m+1}}\,{p_{m+2}}^{t_{m+2}}\dots\,{p_k}^{t_k}} \quad \dots (1)$$

Thus

$${p_{m+1}}^{t_{m+1}}\,{p_{m+2}}^{t_{m+2}}\,\dots\,{p_k}^{t_k} \mid (\omega)^{\varphi({p_{m+1}}^{t_{m+1}}\,{p_{m+2}}^{t_{m+2}}\,\dots\,{p_k}^{t_k})+1} - \omega.$$

Since $s_i \geq t_i$ for $1 \leq i \leq m$, then ${p_1}^{t_1}\,{p_2}^{t_2}\dots\,{p_m}^{t_m} \mid \omega$ which implies

$${p_1}^{t_1}\,{p_2}^{t_2}\dots\,{p_m}^{t_m} \mid (\omega)^{\varphi({p_{m+1}}^{t_{m+1}}\,{p_{m+2}}^{t_{m+2}}\,\dots\,{p_k}^{t_k})+1} - \omega \quad \dots (2)$$

From (1) and (2) we obtain:

$$n \mid (\omega)^{\varphi({p_{m+1}}^{t_{m+1}}\,{p_{m+2}}^{t_{m+2}}\,\dots\,{p_k}^{t_k})+1} - \omega,$$

that is

$$(\omega)^{\varphi({p_{m+1}}^{t_{m+1}}\,{p_{m+2}}^{t_{m+2}}\,\dots\,{p_k}^{t_k})+1} - \omega \equiv 0 \pmod{n}.$$

Hence ω is a trivial semi nilpotent element.■

Lemma 2.4. Let $p_1, p_2, \dots, p_k$ be distinct primes and $\lambda = \ell\text{cm}\,(\varphi(p_1), \varphi(p_2), \dots, \varphi(p_k))$. If $(a)^{p_i-1} \equiv 1 \pmod{p_i}$ for each $1 \leq i \leq k$, then $a^{\lambda} \equiv 1 \pmod{p_1 p_2 \dots p_k}$.

Proposition 2.5. Consider $\mathbb{Z}_n$, with the prime factorization of $n = {p_1}^{t_1}\,{p_2}^{t_2}\dots\,{p_k}^{t_k}$, with at least one of $t_i > 1$ and $0 \neq \omega \in \mathbb{Z}_n$. Then ω is a nontrivial semi nilpotent if ω has one of the forms:

i) $\omega = {p_1}^{s_1}\,{p_2}^{s_2}\,\dots\,{p_k}^{s_k}\,{r_1}^{\alpha_1}{r_2}^{\alpha_2}\dots{r_m}^{\alpha_m}$, such that $s_1, s_2, \dots, s_k$ are different from zero.

ii) $\omega = {q_1}^{s_1}\,{q_2}^{s_2}\dots\,{q_\ell}^{s_\ell}\;\;{r_1}^{\alpha_1}{r_2}^{\alpha_2}\dots{r_m}^{\alpha_m}$ such that for each $1 \leq j \leq \ell$ there exists $1 \leq i \leq k$, $q_j = p_i$ with $\ell < k$ and $0 < s_i < t_i$, for at least one $i \in \{1, 2, \dots, \ell\}$.

In both cases $\alpha_i \geq 0$, r_j are primes distinct from $p_1, p_2, \dots, p_k$

Proof.

i) Since ω is divisible by $p_1, p_2, \dots, p_k$, so ω is a nilpotent element and by Proposition 2.1, ω is a nontrivial semi nilpotent element.

ii) Without loss of generality suppose that $t_1 > 1$ and $\omega = {p_1}^{s_1}\,{p_2}^{s_2}\dots\,{p_\ell}^{s_\ell}\,{r_1}^{\alpha_1}{r_2}^{\alpha_2}\dots{r_m}^{\alpha_m}$ for $0 < s_1 < t_1$, $\ell < k$. By Fermat's Little Theorem, $(\omega)^{p_j-1} \equiv 1 \pmod{p_j}$ for each $\ell + 1 \leq j \leq k$

Put $\lambda = \ell\text{cm}\,(\varphi(p_{\ell+1}), \varphi(p_{\ell+2}), \dots, \varphi(p_k))$. Then by Lemma 2.4,

$$\omega^{\lambda} \equiv 1 \pmod{p_{\ell+1}\,p_{\ell+2}\dots p_k}$$

Hence $\omega^{\lambda+1} \equiv \omega \pmod{p_{\ell+1}\,p_{\ell+2}\dots p_k}$
which means $p_{\ell+1}\,p_{\ell+2}\dots p_k \mid \omega^{\lambda+1} - \omega$
consequently $p_1\,p_2\dots p_k \mid \omega^{\lambda+1} - \omega$, so $\omega^{\lambda+1} - \omega$ is a nilpotent element of $\mathbb{Z}_n$. It remains to show that $\omega^{\lambda+1} - \omega \not\equiv 0 \pmod{n}$.
Now $\omega^{\lambda+1} - \omega = \omega(\omega^{\lambda} - 1)$, then from the fact that ${p_1}^{s_1} \mid \omega$, ${p_1}^{t_1} \nmid \omega$ and $p_1 \nmid \omega^{\lambda} - 1$, we deduce that ${p_1}^{t_1} \nmid \omega^{\lambda+1} - \omega$. So $\omega^{\lambda+1} - \omega \not\equiv 0 \pmod{n}$. Therefore ω is a nontrivial semi nilpotent element of $\mathbb{Z}_n$.■

Proposition 2.6. Every nontrivial nilpotent element in a ring $\mathcal{R}$ is a nontrivial semi nilpotent.

Proof. Clearly every nilpotent element is semi nilpotent. Let $0 \neq \omega$ be a nilpotent element of $\mathcal{R}$ and n be the least positive integer such that $\omega^n = 0$. If $\omega^k - \omega = 0$ for some k, $1 < k < n$, then $\omega^k = \omega$. Now $0 = \omega^n = \omega^k\,\omega^{n-k} = \omega^{n-k+1}$ but $n - k + 1 < n$, contradiction. Hence ω is a nontrivial semi nilpotent.■

Theorem 2.7. Every element in $\mathbb{Z}_2\, G \backslash \{0, 1\}$, where G is a cyclic group of order 2^n and n is an even integer is a nontrivial semi nilpotent element.

Proof. Let $\omega \in \mathrm{Z}_2\, G \backslash \{0, 1\}$. Then ω is of the form $\omega = a_0 + a_1\,\mathrm{g} + \ldots + a_{2^n-1}\mathrm{g}^{2^n-1}$ such that there is $a_i \neq 0$ for some $i > 0$. If ω has an even number of nonzero terms say 2ℓ that is $\omega = a_{i_1}\mathrm{g}^{i_1} + a_{i_2}\,\mathrm{g}^{i_2} + \ldots + a_{i_{2\ell}}\mathrm{g}^{i_{2\ell}}$, then

$$\omega^{2^n} = \underbrace{1 + 1 + \cdots + 1}_{2\ell-times} = 0,$$

which means that ω is a nontrivial nilpotent. Hence by Proposition 2.6, ω is a nontrivial semi nilpotent. If ω has an odd number of nonzero terms say $2\ell + 1$, that is $\omega = a_{i_1}\mathrm{g}^{i_1} + a_{i_2}\,\mathrm{g}^{i_2} + \ldots + a_{i_{2\ell+1}}\mathrm{g}^{i_{2\ell+1}}$, thus

$$\omega^{2^n} = \underbrace{1 + 1 + \cdots + 1}_{(2\ell+1)-times} = 1$$

Hence $\omega^{2^n} - \omega = 1 - \omega = 1 + \omega$, so ω is a nontrivial semi nilpotent element.■

In what follows we answer the following open problem given by Vasantha: Let $\mathbb{Z}_p$ be the prime field of characteristic p $(p > 2)$ and $G = <\mathrm{g} : \mathrm{g}^q = 1>$ be a cyclic group of order q.

a. If $(p,q) = 1$, can the group ring $\mathbb{Z}_p G$ have nontrivial semi nilpotent elements?
b. If $p|q$, can the group ring $\mathbb{Z}_p G$ have nontrivial semi nilpotent elements?

Now, we need the following Lemma's.

Lemma 2.8. In $\mathbb{Z}_p G$, p prime, and $G = <\mathrm{g} : \mathrm{g}^q = 1>$ is a cyclic group of order q, we have:

$$(a_0 + a_1\,\mathrm{g} + a_2\mathrm{g}^2 + \cdots + a_{q-1}\mathrm{g}^{q-1})^p = {a_0}^p + (a_1\,\mathrm{g})^p + \cdots + (a_{q-1}\mathrm{g}^{q-1})^p$$

Moreover by induction one can show that

$$(a_0 + a_1\,\mathrm{g} + a_2\mathrm{g}^2 + \cdots + a_{q-1}\mathrm{g}^{q-1})^{p^k} = {a_0}^{p^k} + (a_1\,\mathrm{g})^{p^k} + \cdots + (a_{q-1}\mathrm{g}^{q-1})^{p^k}$$

Lemma 2.9. For each $a \in \mathbb{Z}_p$ we have $(a)^{p^k} \equiv a \pmod{p}$, for any $k \in \mathbb{N}$.

Theorem 2.10. Let $p_1, p_2, \ldots, p_r$ and p be distinct primes. If G is a cyclic group of order m with the prime factorization of $m = {p_1}^{k_1}{p_2}^{k_2} \ldots {p_r}^{k_r}$, then every element in the group ring $\mathbb{Z}_p G$, is a trivial semi nilpotent element.

Proof. Suppose $\omega = a_0 + a_1\,\mathrm{g} + a_2\mathrm{g}^2 + \cdots + a_{m-1}\mathrm{g}^{m-1} \in \mathbb{Z}_p G$.

We will show that $(\omega)^{p^{\varphi(p_1{}^{k_1}p_2{}^{k_2}\dots p_r{}^{k_r})}} = \omega$.

By Lemma 2.8, we obtain:

$$(\omega)^{p^{\varphi(m)}} = (a_0 + a_1\,\mathrm{g} + a_2\mathrm{g}^2 + \cdots + a_{m-1}\mathrm{g}^{m-1})^{p^{\varphi\left(p_1{}^{k_1}p_2{}^{k_2}\dots p_r{}^{k_r}\right)}}$$

$$= (a_0 + a_1\,\mathrm{g} + \cdots + a_{m-1}\mathrm{g}^{m-1})^{p^{\left(p_1{}^{k_1} - p_1{}^{k_1-1}\right)\left(p_2{}^{k_2} - p_2{}^{k_2-1}\right)\dots\left(p_r{}^{k_r} - p_r{}^{k_r-1}\right)}}$$

$$= (a_0)^{p^{\left(p_1{}^{k_1} - p_1{}^{k_1-1}\right)\left(p_2{}^{k_2} - p_2{}^{k_2-1}\right)\dots\left(p_r{}^{k_r} - p_r{}^{k_r-1}\right)}}$$

$$+ (a_1\mathrm{g})^{p^{\left(p_1{}^{k_1} - p_1{}^{k_1-1}\right)\left(p_2{}^{k_2} - p_2{}^{k_2-1}\right)\dots\left(p_r{}^{k_r} - p_r{}^{k_r-1}\right)}}$$

$$+ \cdots + (a_{m-1}\mathrm{g}^{m-1})^{p^{\left(p_1{}^{k_1} - p_1{}^{k_1-1}\right)\left(p_2{}^{k_2} - p_2{}^{k_2-1}\right)\dots\left(p_r{}^{k_r} - p_r{}^{k_r-1}\right)}} \qquad \dots.(*)$$

By Euler's Theorem, $p^{\varphi(p_1{}^{k_1}p_2{}^{k_2}\dots p_r{}^{k_r})} \equiv 1 \pmod{p_1{}^{k_1}p_2{}^{k_2}\dots p_r{}^{k_r}}$, so $p^{(p_1{}^{k_1} - p_1{}^{k_1-1})\,(p_2{}^{k_2} - p_2{}^{k_2-1})\dots(p_r{}^{k_r} - p_r{}^{k_r-1})} \equiv 1 \pmod{p_1{}^{k_1}p_2{}^{k_2}\dots p_r{}^{k_r}}$, and by **Lemma 2.9**, then:

$$(\omega)^{p^{\varphi(m)}} = a_0 + a_1\,\mathrm{g} + a_2\mathrm{g}^2 + \cdots + a_{m-1}\mathrm{g}^{m-1} = \omega,$$

Consequently, $(\omega)^{p^{(p_1{}^{k_1} - p_1{}^{k_1-1})\,(p_2{}^{k_2} - p_2{}^{k_2-1})\dots(p_r{}^{k_r} - p_r{}^{k_r-1})}} - \omega = 0$.

Therefore, ω is a trivial semi nilpotent element.■

The following corollary is a direct consequence of **Theorem 2.10**.

Corollary 2.11. The group ring $\mathbb{Z}_pG$,where p prime and G is a cyclic group of order q such that $p \nmid q$, has no nontrivial semi nilpotent element.

Theorem 2.12. Let $\mathbb{Z}_p$ be the prime field of characteristic $p\,(p \geq 2)$, and G be a cyclic group of order q. If $p|q$, then the group ring $\mathbb{Z}_pG$ has nontrivial semi nilpotent elements.

Proof. Since $p|q$ then $q = pm$ for some $m \in \mathbb{N}$. Now let

$$\omega = 1 + \mathrm{g}^m + \mathrm{g}^{2m} + \cdots + \mathrm{g}^{(p-1)m} \in \mathbb{Z}_pG.$$

$$(\omega)^p = \left(1 + \mathrm{g}^m + \mathrm{g}^{2m} + \cdots + \mathrm{g}^{(p-1)m}\right)^p$$

$$= 1 + (\mathrm{g}^m)^p + (\mathrm{g}^{2m})^p + \cdots + (\mathrm{g}^{(p-1)m})^p \quad (\textbf{Lemma 2.8})$$

$$= 1 + \mathrm{g}^q + \mathrm{g}^{2q} + \cdots + \mathrm{g}^{(p-1)q} = \underbrace{1 + 1 + \cdots + 1}_{p-times}$$

$$= p\,(1) = 0.$$

So ω is a nontrivial nilpotent element and by Proposition 2.6, ω is a nontrivial semi nilpotent element. Hence the group ring $\mathbb{Z}_pG$ has nontrivial semi nilpotent elements.■

3. Conclusion

In this work the form of trivial and nontrivial semi nilpotent elements obtained in $\mathbb{Z}_n$. It is shown that every element except 0 and 1 in the group ring $\mathbb{Z}_2G$ where G is a cyclic group of order 2^n, for even n, is a nontrivial semi nilpotent element. In addition we have got the answer of the following open problem given by Vasantha: Let $\mathbb{Z}_p$ be the prime field of characteristic $p\,(p > 2)$ and $G = <g: g^q = 1 >$ be a cyclic group of order q.

a. If $(p,q) = 1$, can the group ring $\mathbb{Z}_pG$ have nontrivial semi nilpotent elements?
b. If $p|q$, can the group ring $\mathbb{Z}_pG$ have nontrivial semi nilpotent elements?

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