On the Banach Algebra $\mathscr{B}(c(N^t))$

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Abstract: In this paper, we give some properties of the Banach Algebras of the bounded operators on the BK space $c(N^t)$ which is the Nörlund domain in the convergent sequence space introduced by Tuğ and Başar (2016). We prove that the class $(c(N^t), c(N^t))$ is a Banach algebra with respect to the norm $\|A\| = \|L_A\|$ for all $A \in (c(N^t), c(N^t))$.

Keywords: Norlund Matrix, Sequence Spaces, Banach Algebras, Matrix Norm

1. Preliminaries, Background and Notation

We denote the space of all complex valued sequences by ω. Each vector subspace of ω is called as a *sequence space*, as well. The spaces of all bounded, convergent and null sequences are denoted by ℓ_∞, c and c_0, respectively. By ϕ, we mean the space of all finitely non-zero sequences. A sequence space μ is called an FK-space if it is a complete linear metric space with continuous coordinates $p_n : \mu \to \mathbb{C}$ with $p_n(x) = x_n$ for all $x = (x_n) \in \mu$ and every $n \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$. A normed FK-spaces is called a BK-space, that is, a BK-space is a Banach space with continuous coordinates, (Choudhary & Nanda, 1989, pp. 272-273). The sequence spaces ℓ_∞, c and c_0 are BK-spaces with the usual sup-norm defined by $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|$. By ℓ_1, ℓ_p, cs, cs_0 and bs, we denote the spaces of all absolutely convergent, p-absolutely convergent, convergent, convergent to zero and bounded series, respectively; where $1 < p < \infty$.
The alpha-dual λ^α, beta-dual λ^β and gamma-dual λ^γ of a sequence space λ are defined by

$$\begin{aligned} \lambda^\alpha &:= \{x = (x_k) \in \omega : xy = (x_k y_k) \in \ell_1 \text{ for all } y = (y_k) \in \lambda\}, \\ \lambda^\beta &:= \{x = (x_k) \in \omega : xy = (x_k y_k) \in cs \text{ for all } y = (y_k) \in \lambda\}, \\ \lambda^\gamma &:= \{x = (x_k) \in \omega : xy = (x_k y_k) \in bs \text{ for all } y = (y_k) \in \lambda\}. \end{aligned}$$

Let λ, μ be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $k, n \in \mathbb{N}$. Then, we say that A defines a *matrix transformation* from λ into μ and we denote it by writing $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ; where

$$(Ax)_n = \sum_k a_{nk} x_k \tag{1}$$

provided the series on the right side of (1) converges for each $n \in \mathbb{N}$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if Ax exists, i.e. $A_n \in \lambda^\beta$ for all $n \in \mathbb{N}$ and belongs to μ for all $x \in \lambda$, where A_n denotes the sequence in the n-th row of A.
Let X be a Banach space with the norm $\|.\|_X$. We denote the set of all bounded linear operators, which maps X into itself by $\mathscr{B}(X)$. That is, $A \in \mathscr{B}(X)$ if and only if A is linear and

$$\|A\|^*_{\mathscr{B}(X)} = \sup_{x\neq 0} \frac{\|Ax\|_X}{\|x\|_X} < \infty$$

It is known that $\mathscr{B}(X)$ is a Banach algebra with its norm $\|A\|^*_{\mathscr{B}(X)}$, see Jarrah and Malkowsky (1990). If a normed sequence space λ contains a sequence (b_n) with the following property that for every $x \in \lambda$ there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} \|x - (\alpha_0 b_0 + \alpha_1 b_1 + \cdots + \alpha_n b_n)\| = 0$$

then (b_n) is called a Schauder basis for λ. The series $\sum_k \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum_k \alpha_k b_k$.

If λ is an FK-space, $\phi \subset \lambda$ and (e^k) is a basis for λ then λ is said to have AK property, where e^k is a sequence whose only term in k^{th} place is 1 the others are zero for each $k \in \mathbb{N}$ and $\phi = span\{e^k\}$. If ϕ is dense in λ, then λ is called AD-space, thus AK implies AD. It is also well known that if X has AK then $\mathscr{B}(X) = (X,X)$, see Malkowsky and Al (2003)

2. The Sequence Spaces $c_0(N^t)$ and $c(N^t)$ of Non-absolute Type

Let (t_k) be a nonnegative real sequence with $t_0 > 0$ and $T_n = \sum_{k=0}^n t_k$ for all $n \in \mathbb{N}$. Then, the Nörlund mean with respect to the sequence $t = (t_k)$ is defined by the matrix $N^t = (a^t_{nk})$ as follows

$$a^t_{nk} = \begin{cases} \frac{t_{n-k}}{T_n} & , \quad 0 \le k \le n, \\ 0 & , \quad k > n \end{cases}$$

for every $k,n \in \mathbb{N}$. It is known that the Nörlund matrix N^t is regular if and only if $t_n/T_n \to 0$, as $n \to \infty$ (Hardy, 2000, Theorem 16, p. 64), and is reduced in the case $t = e = (1,1,1,\ldots)$ to the matrix C_1 of arithmetic mean. Additionally, for $t_n = A_n^{r-1}$ for all $n \in \mathbb{N}$, the method N^t is reduced to the Cesàro method C_r of order $r > -1$, where

$$A^r_n = \begin{cases} \frac{(r+1)(r+2)\cdots(r+n)}{n!} & , \quad n = 1,2,3,\ldots, \\ 1 & , \quad n = 0. \end{cases}$$

Let $t_0 = D_0 = 1$ and define D_n for $n \in \{1,2,3,\ldots\}$ by

$$D_n = \begin{vmatrix} t_1 & 1 & 0 & 0 & \cdots & 0 \\ t_2 & t_1 & 1 & 0 & \cdots & 0 \\ t_3 & t_2 & t_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & t_{n-4} & \cdots & 1 \\ t_n & t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_1 \end{vmatrix}.$$

Then, the inverse matrix $U^t = (u^t_{nk})$ of Nörlund matrix N^t was defined by Mears (1943) for all $n \in \mathbb{N}$, as follows;

$$u^t_{nk} = \begin{cases} (-1)^{n-k} D_{n-k} T_k & , \quad 0 \le k \le n, \\ 0 & , \quad k > n. \end{cases}$$

Additionally, the inverse of Nörlund matrix and some multiplication theorems for Nörlund mean were studied by Mears (1943); Wang (1978).
The domain λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A = \{x = (x_k) \in \omega : Ax \in \lambda\}$$

which is a sequence space. The domain of Nörlund matrix N^t in the classical sequence spaces ℓ_∞ and ℓ_p were introduced by Wang (1978), where $1 \le p < \infty$.
Tug and Basar (2016) introduced the sequence spaces $c_0(N^t)$ and $c(N^t)$ as the set of all sequences whose N^t-transforms are in the spaces of null and convergent sequences, respectively, that is

$$\begin{aligned} c_0(N^t) &:= \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \frac{1}{T_n} \sum_{k=0}^{n} t_{n-k} x_k = 0 \right\}, \\ c(N^t) &:= \left\{ x = (x_k) \in \omega : \exists l \in \mathbb{C} \text{ such that } \lim_{n\to\infty} \frac{1}{T_n} \sum_{k=0}^{n} t_{n-k} x_k = l \right\}. \end{aligned}$$

They defined the sequence $y = (y_k)$ by the N^t-transform of a sequence $x = (x_k)$, that is,

$$y_k = (N^t x)_k = \frac{1}{T_k} \sum_{j=0}^{k} t_{k-j} x_j \tag{2}$$

for all $k \in \mathbb{N}$. Therefore, by applying U^t to the sequence y defined by (2) we obtain that

$$x_k = (U^t y)_k = \sum_{j=0}^{k} (-1)^{k-j} D_{k-j} T_j y_j \tag{3}$$

for all $k \in \mathbb{N}$. Throughout the text, we suppose that the terms of the sequences $x = (x_k)$ and $y = (y_k)$ are connected with the relation (2 and 3).

Theorem 2.1. *(Tug and Basar (2016)) The sequence spaces $c_0(N^t)$ and $c(N^t)$ are the linear spaces with the co-ordinatewise addition and scalar multiplication which are the BK-spaces with the norm*

$$\|x\|_{c_0(N^t)} = \|x\|_{c(N^t)} = \|N^t x\|_\infty = \sup_n \left(\frac{1}{T_n} \sum_{k=0}^{n} t_{n-k} |x_k| \right) \tag{4}$$

Theorem 2.2. *(Tug and Basar (2016)) Let $\alpha_k = (N^t x)_k$ for all $k \in \mathbb{N}$. Define the sequence $\{u^{(n)}\} = \{u_k^{(n)}\}_{k\in\mathbb{N}}$ in the space $c_0(N^t)$ by*

$$u_k^{(n)} = \begin{cases} (-1)^{n-k} D_{n-k} T_k &, \quad 0 \le k \le n, \\ 0 &, \quad k > n \end{cases}$$

for every fixed $n \in \mathbb{N}$.

(a) *The sequence $\{u^{(n)}\}_{n\in\mathbb{N}}$ is a basis for the space $c_0(N^t)$ and any $x \in c_0(N^t)$ has a unique representation of the form $x = \sum_{k=0}^{\infty} \alpha_k u_k^n$.*

(b) *The set $\{e, u^{(n)}\}$ is a basis for the sequence space $c(N^t)$ and any $x \in c(N^t)$ has a unique representation of the form $x = le + \sum_{k=0}^{\infty} (\alpha_k - l) u_k^n$, where $l = \lim_{k\to\infty} \alpha_k$.*

Theorem 2.3. *(Tug and Basar (2016)) Define the set d_2^t, as follows;*

$$d_2^t := \left\{ a = (a_k) \in \omega : \sup_{n\in\mathbb{N}} \sum_k \left| \sum_{j=k}^{n} (-1)^{j-k} D_{j-k} T_k a_j \right| < \infty \right\}.$$

Then, $\{c_0(N^t)\}^\beta = \{c(N^t)\}^\beta = d_2^t \cap cs$.

Theorem 2.4. *(Tug and Basar (2016)) $A = (a_{nk}) \in (c(N^t) : c)$ if and only if*

$$A_n \in \{c(N^t)\}^\beta \text{ for each } n \in \mathbb{N},$$
$$F \in (c : c).$$

where we define the matrix $F = (f_{nk})$ via multiplication of the matrices A and N^t by the products AN^t, that is

$$f_{nk} := \sum_{j=k}^{\infty} (-1)^{j-k} D_{j-k} T_k a_{nj}$$

for all $k, n \in \mathbb{N}$.

3. The Banach Algebra $\mathscr{B}(c(N^t))$

In this section, we show that $\mathscr{B}(c(N^t))$ is Banach algebra with respect to the norm $\|.\|$ defined by (4) for all $A \in (c(N^t), c(N^t))$. Since $c(N^t)$ has AK, we have $\mathscr{B}(c(N^t)) = (c(N^t), c(N^t))$. So $A \in \mathscr{B}(c(N^t))$ if and only if $A \in (c(N^t), c(N^t))$ and we have

$$\|A\|_{\mathscr{B}(c(N^t))} = \sup_{x \neq 0} \left(\frac{\|Ax\|_{c(N^t)}}{\|x\|_{c(N^t)}} \right) < \infty$$

Definition 3.1. *(Conway (2013)) An algebra over $\mathbb{F}$ is a vector space $\mathscr{A}$ over $\mathbb{F}$ such that $x, y \in \mathscr{A}$ with a unique product $x.y \in \mathscr{A}$ is defined with the properties*

(i) $(xy)z = x(yz)$,

(ii) $x(y+z) = xy + xz$,

(iii) $(x+y)z = xz + yz$,

(iv) $\alpha(xy) = (\alpha x)y = x(\alpha y)$

for all $x, y, z \in \mathscr{A}$ and $\alpha \in \mathbb{F}$.

Then the following immediate notations can be stated. $\mathscr{A}$ is called commutative(or abelian) if $\forall x, y \in \mathscr{A}$, $xy = yx$. $\mathscr{A}$ is called an algebra with identity if $\mathscr{A}$ contains an element e such that $\forall x \in \mathscr{A}$, $ex = xe = x$, this e is called identity.

Definition 3.2. *(Conway (2013)) A Banach algebra $\mathscr{A}$ is a normed space which is an algebra such that for all $x, y \in \mathscr{A}$*

$$\|xy\| \leq \|x\| \|y\|$$

and if $\mathscr{A}$ has an identity e, then $\|e\| = 1$.

Now, we state the following significant lemma to define and prove the sufficient conditions of Banach algebra $\mathscr{B}(c(N^t))$.

Lemma 3.1. *(a) The matrix product $B.A$ is defined for all $A, B \in (c(N^t), c(N^t))$; essentially*

$$\sum_{m=0}^{\infty} |b_{nm} a_{mk}| \leq \|B_n\|_{c(N^t)} \|A^k\| \text{ for all } n \text{ and } k.$$

(b) Matrix multiplication is associative in $(c(N^t), c(N^t))$.

(c) The space $(c(N^t), c(N^t))$ *is a Banach space with respect to the norm*

$$\|A\| = \sup_n \left(\frac{1}{T_n} \sum_{k=0}^{n} t_{n-k} \left| \sum_{j=0}^{k} a_{nj} \right| \right)$$

Proof. (a) Let $A, B \in (c(N^t), c(N^t))$. Since for all $x \in c(N^t)$ it satisfies that $Ax \in c(N^t)$. So specifically $e^{(k)} \in c(N^t)$ implies that

$$Ae^{(k)} = (A_i e^{(k)})_{i=0}^{\infty} = (a_{ik})_{i=0}^{\infty} = A^k \in c(N^t), \quad for\ all\ k \in \mathbb{N}.$$

Thus we have

$$\|A^k\| = \sup_n \left(\frac{1}{T_n} \sum_{k=0}^{n} t_{n-k} \left| \sum_{i=0}^{\infty} a_{ik} \right| \right) < \infty, \quad for\ all\ \ k \in \mathbb{N}.$$

Furthermore $B \in (c(N^t), c(N^t))$ implies $B_n \in \{c(N^t)\}^{\beta}$ for all $n \in \mathbb{N}$. Therefore

$$\|B_n\|_{c(N^t)} = \sup_{n\in\mathbb{N}} \sum_k \left| \sum_{j=k}^{n} (-1)^{j-k} D_{j-k} T_k \left(\sum_{i=0}^{j} b_{ni} \right) \right| < \infty, \quad for\ all\ \ n \in \mathbb{N}.$$

Now we have the following by (3) and (3) that

$$\begin{aligned}
|B_n A^k| \le \sum_{i=0}^{\infty} |b_{ni} a_{ik}| &= \sum_{i=0}^{\infty} \left| \sum_{i=k}^{n} (-1)^{i-k} D_{i-k} T_k b_{ni} . \frac{1}{T_n} \sum_{k=0}^{n} t_{n-k} a_{ik} \right| \\
&\le \sum_k \left| \left(\sum_{j=k}^{n} (-1)^{j-k} D_{j-k} T_k \left(\sum_{i=0}^{j} b_{ni} \right) \right) . \left(\frac{1}{T_n} \sum_{k=0}^{n} t_{n-k} \left(\sum_{i=0}^{\infty} a_{ik} \right) \right) \right| \\
&\le \sum_k \left| \sum_{j=k}^{n} (-1)^{j-k} D_{j-k} T_k \left(\sum_{i=0}^{j} b_{ni} \right) \right| . \sup_n \frac{1}{T_n} \sum_{k=0}^{n} t_{n-k} \left| \sum_{i=0}^{\infty} a_{ik} \right| \\
&= \|B_n\|_{c(N^t)} \|A^k\| < \infty, \text{ for all } n, k.
\end{aligned}$$

(b) Let $A, B, C \in (c(N^t), c(N^t))$. We will show that the series $\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} a_{nm} b_{mk} c_{kj}$ is N^t-convergent for all n and j. It can be easily shown since Nörlund matrix is a triangular matrix. We omit the details.
(c) Now we will show that the space $(c(N^t), c(N^t))$ is a Banach space. We assume that $(A^{(i)})_{i=0}^{\infty}$ is a Cauchy sequence in $(c(N^t), c(N^t))$ and the space $c(N^t)$ has AK property, then it is a Cauchy sequence in $\mathscr{B}(c(N^t), c(N^t))$. There is $L_A \in \mathscr{B}(c(N^t), c(N^t))$ with $L_{A^{(i)}} \to L_A$. Since $c(N^t)$ has AK property then there is a matrix $A \in (c(N^t), c(N^t))$ such that $Ax = L_A(x)$ for all $x \in c(N^t)$. It shows that there exists $M \in \mathbb{N}$ such that

$$\|A^{(i)} - A^{(l)}\|_{c(N^t)} = \sup_n \left(\frac{1}{T_n} \sum_{k=0}^{n} t_{n-k} \left| \sum_{j=0}^{k} (a_{nj}^{(i)} - a_{nj}^{(l)}) \right| \right) < \frac{\varepsilon}{2}, \quad for\ all\ i, l \ge M. \tag{5}$$

So, $A^{(i)}$ is a Cauchy sequence in the space $c(N^t)$ which is complete normed space. Then there is a matrix $A \in (c(N^t), c(N^t))$ such that $Ax = L_A(x)$ for all $x \in c(N^t)$

$$\|A^{(i)} - A\|_{c(N^t)} = \sup_n \left(\frac{1}{T_n} \sum_{k=0}^{n} t_{n-k} \left| \sum_{j=0}^{k} (a_{nj}^{(i)} - a_{nj}) \right| \right) < \frac{\varepsilon}{2} \tag{6}$$

we we run the equalities (5) and (6) we will see that $A^{(j)} \to A \in (c(N^t), c(N^t))$. Moreover,

$$\begin{aligned}
\|A\|_{c(N^t)} &= \sup_n \left(\frac{1}{T_n} \sum_{k=0}^{n} t_{n-k} \left| \sum_{j=0}^{k} a_{nj} \right| \right) \\
&= \sup_n \left(\frac{1}{T_n} \sum_{k=0}^{n} t_{n-k} \left| \sum_{j=0}^{k} \left(a_{nj} + a_{nj}^{(i)} - a_{nj}^{(i)} \right) \right| \right) \\
&\leq \sup_n \left(\frac{1}{T_n} \sum_{k=0}^{n} t_{n-k} \left| \sum_{j=0}^{k} \left(a_{nj}^{(i)} - a_{nj} \right) \right| \right) + \sup_n \left(\frac{1}{T_n} \sum_{k=0}^{n} t_{n-k} \left| \sum_{j=0}^{k} a_{nj}^{(i)} \right| \right) < \infty.
\end{aligned}$$

So, $A \in \mathscr{B}(c(N^t), c(N^t))$. This completes the proof. □

Theorem 3.2. *The set $\mathscr{B}(c(N^t)) = (c(N^t), c(N^t))$ is a Banach algebra with the identity and we have*

$$\|Ax\|_{c(N^t)} \leq \|A\|_{\mathscr{B}(c(N^t))} \|x\|_{c(N^t)}, \quad \forall x \in c(N^t).$$

Proof. We should show here that $(c(N^t), c(N^t))$ is complete and if $A, B \in (c(N^t), c(N^t))$, then $A.B \in (c(N^t), c(N^t))$. So these facts obtained as an immediate consequence of Lemma 3.1 by considering (a) and (c). □

Theorem 3.3. *The class $(c_0(N^t), c_0(N^t))$ is a Banach algebra with $\|A\| = \|L_A\|$.*

Proof. To prove this theorem we should show that (i) the class $(c_0(N^t), c_0(N^t))$ is complete and (ii) $B.A \in (c_0(N^t), c_0(N^t))$ where $A, B \in (c_0(N^t), c_0(N^t))$. The proof of (i) can be easily shown by Lemma 3.1(c) with the inclusion $(c_0(N^t), c_0(N^t)) \subset (c(N^t), c(N^t))$. Moreover, the proof of (ii) can be obtained from the Lemma 3.1 by considering (a). □

4. Conclusion

De Malafosse (2004) studied some topological properties of the Banach algebras of bounded operators $\mathscr{B}(l_p(\alpha))$ for $1 \leq p < \infty$, where $l_p(\alpha) = (1/\alpha)^{-1} * l_p$. He also studied the Banach algebras of the bounded operators $\mathscr{B}(X)$, where X is a $BK-$space in de Malafosse (2005). Moreover Malkowsky (Malkowsky (2011); Malkowsky and Djolović (2013)) studied the Banach algebra of matrix transformation between some sequence spaces.
In this work, we study the Banach algebra $\mathscr{B}(c(N^t)) = (c(N^t), c(N^t))$ where $c(N^t)$ is the set of all convergent sequences derived by Nörlund mean which was defined by Tuğ and Başar (Tug and Basar (2016)).

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