Almost and Strongly Almost $B(\tilde{r},\tilde{s},\tilde{t},\tilde{u})$–Summable Double Sequences

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Abstract: In this paper, we define some new almost and strongly almost convergent double sequence spaces $\tilde{B}(\mathcal{C}_f)$, $\tilde{B}(\mathcal{C}_f)$, $\tilde{B}[\mathcal{C}_f]$ and $\tilde{B}[\mathcal{C}_f]$ derived by the domain of four-dimensional sequential band matrix $B(\tilde{r},\tilde{s},\tilde{t},\tilde{u})$ in the spaces $\mathcal{C}_f$, $\mathcal{C}_f$, $[\mathcal{C}_f]$ and $[\mathcal{C}_f]$, respectively. Then we study some topological properties and prove some strict inclusion relations. Moreover, we calculate the $\alpha$, $\beta bp$– and $\gamma$–duals of the new spaces. Finally, we state some known lemmas concerning the four-dimensional matrix classes of almost convergent double sequences, then we characterize some new four-dimensional matrix transformations from and into the new sequence spaces $\tilde{B}(\mathcal{C}_f)$ and $\tilde{B}[\mathcal{C}_f]$. We conclude the paper with several significant results.

Keywords: Four-dimensional band matrix; matrix domain; almost convergence; double sequences; dual spaces; matrix transformations.

1. Introduction and Preliminaries

By the set $\Omega := \{x = (x_{mn}) : x_{mn} \in \mathbb{C}, \forall m, n \in \mathbb{N}\}$, we denote all complex valued double sequences, where $\mathbb{C}$ is the complex field. $\Omega$ is a vector space with coordinatewise addition and scalar multiplication. Any vector subspace of $\Omega$ is called a double sequence space. $\mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_bp$, $\mathcal{C}_r$, and $\mathcal{L}_q$ denote the classical spaces of all double sequences that are bounded, convergent in the Pringsheim sense, convergent in the Pringsheim sense and bounded, regular convergent, and $q$-absolutely summable, respectively, where $0 < q < \infty$. It is well known that the space $\mathcal{L}_q$ becomes the space $\mathcal{L}_u$ in the case $q = 1$. Moreover, by $\mathcal{B}\mathcal{P}$, $\mathcal{C}\mathcal{P}$, $\mathcal{P}$, where $\mathcal{P} = \{p, bp, r\}$, we denote all bounded and $\mathcal{P}$-convergent series, respectively.

Let $E$ be any double sequence space. Then,

$$E^{B(\tilde{\varnothing})} := \left\{ a = (a_{kl}) \in \Omega : \{a_{kl}x_{kl}\} \in \mathcal{C}\mathcal{P}, \text{for every } x = (x_{kl}) \in E \right\},$$

$$E^{\alpha} := \left\{ a = (a_{kl}) \in \Omega : \{a_{kl}x_{kl}\} \in \mathcal{L}_u, \text{for every } x = (x_{kl}) \in E \right\},$$

$$E^{\tilde{\varnothing}} := \left\{ a = (a_{kl}) \in \Omega : \{a_{kl}x_{kl}\} \in \mathcal{B}\mathcal{P}, \text{for every } x = (x_{kl}) \in E \right\}.$$

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Therefore, let $E_1$ and $E_2$ are arbitrary double sequences with $E_2 \subset E_1$ then the inclusions $E_1^\alpha \subset E_2^\alpha$, $E_1^\gamma \subset E_1^\alpha$ and $E_1^{\beta(\vartheta)} \subset E_1^\alpha$ hold. But the inclusion $E_1^\gamma \subset E_1^{\beta(\vartheta)}$ does not hold, since $E_p \setminus M_u$ is not empty.

Let $A = (a_{mnkl})_{m,n,k,l \in \mathbb{N}}$ be an infinite four-dimensional matrix and $E_1, E_2 \in \Omega$. We write

$$y_{mn} = A_{mn}(x) = \vartheta - \sum_{k,l} a_{mnkl}x_{kl} \text{ for each } m,n \in \mathbb{N}.$$  

(1)

We say that $A$ defines a matrix transformation from $E_1$ to $E_2$ if

$$A(x) = (A_{mn}(x))_{m,n \in \mathbb{N}} \text{ for all } x \in E_1.$$  

(2)

The $\vartheta$–summability domain $E_A^{(\vartheta)}$ of a four-dimensional infinite matrix $A$ in a double sequence space $E$ is defined by

$$E_A^{(\vartheta)} = \left\{ x = (x_{kl}) \in \Omega : Ax = (\vartheta - \sum_{k,l} a_{mnkl}x_{kl})_{m,n \in \mathbb{N}} \text{ exists and is in } E \right\},$$

which is a sequence space. The above notation (2) says that $A = (a_{mnkl})_{m,n,k,l \in \mathbb{N}}$ maps the space $E_1$ into the space $E_2$ if $E_1 \subset (E_2)_A^{(\vartheta)}$ and we denote the set of all four-dimensional matrices that map the space $E_1$ into the space $E_2$ by $(E_1 : E_2)$. Thus, $A \in (E_1 : E_2)$ if and only if the double series on the right side of (2) $\vartheta$–converges for each $m,n \in \mathbb{N}$, i.e., $A_{mn} \in (E_1)^{\beta(\vartheta)}$ for all $m,n \in \mathbb{N}$ and we have $Ax \in E_2$ for all $x \in E_1$.

Adams (1933) defined that the four-dimensional infinite matrix $A = (a_{mnkl})$ is a triangular matrix if $a_{mnkl} = 0$ for $k > m$ or $l > n$ or both. We also say by Adams (1933) that a triangular matrix $A = (a_{mnkl})$ is called a triangle if $a_{mnmn} \neq 0$ for all $m,n \in \mathbb{N}$. One can be observed easily that if $A$ is triangle, then $E_A^{(\vartheta)}$ and $E$ are linearly isomorphic.

The concept of almost convergence for single sequence introduced by Lorentz (1948) and then Moricz and Rhoades (1988) extended the idea of almost convergence for double sequence. He stated that a double sequence $x = (x_{kl})$ of complex numbers is called almost convergent to a generalized limit $L$ if

$$p - \lim_{q,q' \to \infty} \sup_{m,n \geq 0} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} x_{kl} - L \right| = 0.$$ 

In this case, $L$ is called the $f_2$–limit of the double sequence $x$. Then Başarır (1995) defined the concept of strongly almost convergence of double sequences. A double sequence $x = (x_{kl})$ of real numbers is said to be strongly almost convergent to a limit $L_1$ if

$$p - \lim_{q,q' \to \infty} \sup_{m,n \geq 0} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} |x_{kl} - L_1| \right| = 0,$$

and it is uniform in $m,n \in \mathbb{N}$. Now we may define the set of all almost convergent, almost null, strongly almost convergent and strongly almost null double sequences, respectively, as follow;

$$E_f := \left\{ x = (x_{kl}) \in \Omega : \exists L \in \mathbb{C} \ni \lim_{m,n \to \infty} \sup_{m,n \geq 0} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} x_{kl} - L \right| = 0, \text{ uniformly in } m,n \in \mathbb{N} \text{ for some } L \right\}.$$
\begin{align*}
\mathcal{E}_f &:= \ \ \ \ \left\{ x = (x_{kl}) \in \Omega : \exists L \in \mathbb{C} \ni p - \lim_{q,q' \to \infty} \sup_{m,n \geq 0} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{l+n'q} x_{kl} - L \right| = 0, \right. \\
& \quad \quad \text{uniformly in } m,n \in \mathbb{N} \right\}, \\
\mathcal{E}_{f_0} &:= \ \ \ \left\{ x = (x_{kl}) \in \Omega : \exists L \in \mathbb{C} \ni p - \lim_{q,q' \to \infty} \sup_{m,n \geq 0} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{l+n'q} x_{kl} - L \right| = 0, \\
& \quad \quad \text{uniformly in } m,n \in \mathbb{N} \right\}, \\
\mathcal{E}_{f_0} &:= \ \ \ \left\{ x = (x_{kl}) \in \Omega : \exists L \in \mathbb{C} \ni p - \lim_{q,q' \to \infty} \sup_{m,n \geq 0} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{l+n'q} x_{kl} - L \right| = 0, \\
& \quad \quad \text{uniformly in } m,n \in \mathbb{N} \right\}.
\end{align*}

Here we can say for this case that \( L_1 \) is called \([f_2] - \text{limit of a double sequence } x = (x_{kl}) \) and written shortly as \([f_2] - \lim x = L_1 \).

Here we state some geometrical and topological properties of these sets. Unlike single sequence convergent double sequence need not be almost convergent. But it is well known that every bounded convergent double sequence is also almost convergent and every almost convergent double sequence is bounded. That is, the inclusions \( \mathcal{E}_{bp} \subset \mathcal{E}_f \subset \mathcal{A} \) strictly hold. Since the following inequality

\[
\sup_{m,n \geq 0} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{l+n'q} x_{kl} - L \right| \leq \sup_{m,n \geq 0} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{l+n'q} x_{kl} - L \right|.
\]

holds, we can easily say that if a double sequence is strongly almost convergent, that is, the right hand side of the above inequality approaches to zero if we pass to limit as \( q,q' \to \infty \), then the left hand side of the inequality also tends to zero. It says that the inclusion \( [\mathcal{E}_f] \subset \mathcal{E}_f \) holds and it easily can be seen that the double sequence \( x_{kl} = (-1)^k \), for all \( k \in \mathbb{N} \), is in \( \mathcal{E}_f \setminus [\mathcal{E}_f] \). So the inclusion is strictly hold.

Now, we can mention here that the inclusions \( \mathcal{E}_{bp} \subset [\mathcal{E}_{f_0}] \subset [\mathcal{E}_f] \subset \mathcal{E}_{f_0} \subset \mathcal{E}_f \subset \mathcal{A} \) are strictly hold and each inclusion is proper.

Furthermore, the sets \( \mathcal{E}_f \) and \( \mathcal{E}_{f_0} \) are Banach spaces with the norm

\[
\| x \|_{\mathcal{E}_f} = \sup_{q,q',m,n \in \mathbb{N}} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{l+n'q} x_{kl} \right|.
\]

and the sets \( [\mathcal{E}_f] \) and \( [\mathcal{E}_{f_0}] \) are Banach spaces with the norm

\[
\| x \|_{[\mathcal{E}_f]} = \sup_{q,q',m,n \in \mathbb{N}} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{l+n'q} x_{kl} \right|.
\]

The four-dimensional sequential band matrix \( B(\tilde{r},\tilde{s},\tilde{t},\tilde{u}) = \{ b_{mnkl}(\tilde{r},\tilde{s},\tilde{t},\tilde{u}) \} \) was defined and studied by Tuğ, Rakočević, and Malkowsky (2020) as follows; let \( \tilde{r} = (r_m)_{m=0}^\infty \), \( \tilde{s} = (s_m)_{m=0}^\infty \), \( \tilde{t} = (t_n)_{n=0}^\infty \), and \( \tilde{u} = (u_n)_{n=0}^\infty \) be given sequences of real numbers in the set \( c \setminus c_0 \). Then,

\[
b_{mnkl}(\tilde{r},\tilde{s},\tilde{t},\tilde{u}) := \begin{cases} 
-1 & \text{if } (k,l) = (m,n), \\
1 & \text{if } (k,l) = (m,n-1), \\
0 & \text{if } (k,l) = (m-1,n), \\
0 & \text{if } (k,l) = (m-1,n-1), \\
0 & \text{elsewhere}, 
\end{cases}
\]

for all \( m,n,k,l \in \mathbb{N} \). Therefore, the four-dimensional sequential band matrix \( B(\tilde{r},\tilde{s},\tilde{t},\tilde{u}) \)-transforms a double sequence \( x = (x_{mn}) \) into the double sequence \( y = (y_{mn}) \) as follow;

\[
y_{mn} := \{ B(\tilde{r},\tilde{s},\tilde{t},\tilde{u}) x \}_{mn} = \sum_{k,l} b_{mnkl}(\tilde{r},\tilde{s},\tilde{t},\tilde{u}) x_{kl} \quad (3)
\]

\[
y_{mn} = s_{m-1} u_{n-1} x_{m-1,n-1} + s_{m-1} u_n x_{m-1,n} + r_{m-1} u_{n-1} x_{m,n-1} + r_{m-1} u_n x_{m,n}.
\]
for all \(m,n \in \mathbb{N}\). With respect to the above equation in (3), Tuğ et al. (2020) calculated the inverse \(B^{-1}(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u}) = F(\tilde{r}, \tilde{s}, \tilde{t}, u) = \{f_{nmk}(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u})\}\) of \(B(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u})\) which is defined by

\[
f_{nmk}(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u}) := \frac{1}{r_{mn}} \prod_{l=0}^{k-1} \prod_{j=l}^{n-1} \left(\frac{-s_l}{r_j}\right) \left(\frac{-u_j}{t_j}\right) y_{m-k,n-l}, \quad \text{for all } m,n,k,l \in \mathbb{N}.
\]

(4)

for all \(m,n,k,l \in \mathbb{N}\). Thus, by considering the matrix \(B(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u})\) and its inverse matrix \(F(\tilde{r}, \tilde{s}, \tilde{t}, u)\), and keeping the relation between \(x = (x_{mn})\) and \(y = (y_{mn})\) in mind, we can observe the following:

\[
x_{mn} = \frac{1}{r_{mn}} \sum_{k,l=0}^{n} \prod_{i=0}^{m-k-1} \prod_{j=0}^{n-l-1} \left(\frac{-s_i}{r_j}\right) \left(\frac{-u_j}{t_j}\right) y_{m-k,n-l}, \quad \text{for all } m,n \in \mathbb{N}.
\]

(5)

The double sequence spaces \(\tilde{B}(E)\), where \(E = \{c_p, \mathcal{M}_u, \mathcal{C}_c, \mathcal{L}_q\}\) and \(1 \le q < \infty\) were introduced and studied by Tuğ et al. (2020).

**Remark 0.1.** Note that in the case \(s_m = s, r_m = r\) for all \(m \in \mathbb{N}\) and \(t_n = t, u_n = u\) for all \(n \in \mathbb{N}\), the four-dimensional sequential band matrix \(B(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u})\) is reduced to the four-dimensional generalized difference matrix \(B(r,s,t,u)\) which was defined by Tuğ and Başar (2016), and studied in Tuğ (2017a), Tuğ (2017b), Tuğ (2017c), Tuğ (2017d), and Tuğ (2019). Moreover, in the case \(s_m = -r_m = 1\) for all \(m \in \mathbb{N}\) and \(t_n = -u_n = 1\) for all \(n \in \mathbb{N}\), the four-dimensional sequential band matrix \(B(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u})\) is reduced to the four-dimensional difference matrix \( \Delta(1, -1, 1, -1) \) (see Çapan and Başar (2019)). Therefore, the results produced by domain of the matrix \(B(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u})\) are more comprehensive than the corresponding consequences of the matrix domain of the matrices \(B(r,s,t,u)\) and \(\Delta(1, -1, 1, -1)\).

**2. The new spaces of almost and strongly almost convergent double sequences**

In this section, we define spaces \(\tilde{B}(\mathcal{C}_f), \tilde{B}(\mathcal{C}_f), \tilde{B}(\mathcal{C}_f)\) and \(\tilde{B}(\mathcal{C}_f)\) whose four-dimensional sequential band matrix \(B(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u})\) domains are in the double sequence spaces \(\mathcal{C}_f, \mathcal{C}_f, \mathcal{C}_f\) and \(\mathcal{C}_f\), respectively. Then we give some topological properties and prove some strict inclusion relations.

Now, we define the spaces \(\tilde{B}(\mathcal{C}_f), \tilde{B}(\mathcal{C}_f), \tilde{B}(\mathcal{C}_f)\) and \(\tilde{B}(\mathcal{C}_f)\) as follows;

\[
\tilde{B}(\mathcal{C}_f) := \left\{ x = (x_{kl}) \in \Omega : \exists L \in \mathbb{C} : \exists p \geq \lim_{q,d \rightarrow \infty} \sup_{m,n > 0} \frac{1}{(q+1)(q+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q} (\tilde{B}x)_{kl} - L = 0, \right. \]

\[
\left. \text{uniformly in } m,n \in \mathbb{N} \text{ for some } L \right\},
\]

\[
\tilde{B}(\mathcal{C}_f) := \left\{ x = (x_{kl}) \in \Omega : \exists L \in \mathbb{C} : \exists p \geq \lim_{q,d \rightarrow \infty} \sup_{m,n > 0} \frac{1}{(q+1)(q+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q} (\tilde{B}x)_{kl} = 0, \right. \]

\[
\left. \text{uniformly in } m,n \in \mathbb{N} \right\},
\]

\[
\tilde{B}(\mathcal{C}_f) := \left\{ x = (x_{kl}) \in \Omega : \exists L \in \mathbb{C} : \exists p \geq \lim_{q,d \rightarrow \infty} \sup_{m,n > 0} \frac{1}{(q+1)(q+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q} (\tilde{B}x)_{kl} = 0, \right. \]

\[
\left. \text{uniformly in } m,n \in \mathbb{N} \text{ for some } L \right\},
\]

\[
\tilde{B}(\mathcal{C}_f) := \left\{ x = (x_{kl}) \in \Omega : \exists L \in \mathbb{C} : \exists p \geq \lim_{q,d \rightarrow \infty} \sup_{m,n > 0} \frac{1}{(q+1)(q+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q} (\tilde{B}x)_{kl} = 0, \right. \]

\[
\left. \text{uniformly in } m,n \in \mathbb{N} \right\},
\]

where \(\tilde{B}x_{kl} = s_{k,l}u_{l-1}x_{k,j-1} + s_{k,l}u_{l-1}x_{k,j-1} + r_{k,l}u_{l-1}x_{k,j-1} + r_{k,l}t_{l-1}x_{k,j-1} + r_{k,l}t_{l-1}x_{k,j-1}\).

**Theorem 0.1.** The double sequence spaces \(\tilde{B}(\mathcal{C}_f), \tilde{B}(\mathcal{C}_f), \tilde{B}(\mathcal{C}_f)\) and \(\tilde{B}(\mathcal{C}_f)\) are linearly isomorphic to the spaces \(\mathcal{C}_f, \mathcal{C}_f, \mathcal{C}_f\) and \(\mathcal{C}_f\), respectively; that is, \(\tilde{B}(\mathcal{C}_f) \cong \mathcal{C}_f, \tilde{B}(\mathcal{C}_f) \cong \mathcal{C}_f, \tilde{B}(\mathcal{C}_f) \cong \mathcal{C}_f\) and \(\tilde{B}(\mathcal{C}_f) \cong \mathcal{C}_f\).
Proof. Since the other cases can be proved similarly, we prove only \( \tilde{B}(\mathcal{E}_f) \cong \mathcal{E}_f \). To prove this, we need to show the existence of a linear bijection between the spaces \( \tilde{B}(\mathcal{E}_f) \) and \( \mathcal{E}_f \). Let us define the transformation \( T \) from \( \tilde{B}(\mathcal{E}_f) \) to \( \mathcal{E}_f \) by \( x \mapsto Tx = y = B(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u})x \). Linearity of \( T \) is clear. Moreover, it can be seen that \( x = \theta \) whenever \( Tx = \theta \), which means that \( T \) is injective. Let us suppose that an arbitrary \( y = (y_{kl}) \in \mathcal{E}_f \) and define \( x = (x_{mn}) \) via the sequence \( y \) by the relation (5) for all \( m, n \in \mathbb{N} \). Therefore, we obtain by considering the qualities (3) and (5) that

\[
\begin{align*}
\{B(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u})x\}_{mn} &= s_{m-1}u_{l-1}x_{m-1,n-1} + s_{m-1}x_{m-1,n-1} + r_{m}u_{n-1}x_{m,n-1} + r_{m}u_{n}x_{mn} \\
&= s_{m-1}u_{l-1} \left( \prod_{k=0}^{m-1} \prod_{i=0}^{m-k-1} \prod_{j=n-l}^{k} \left( -\frac{s_j}{r_i} \right) \left( -\frac{u_j}{t_j} \right) y_{m-k-1,n-l-1} \right) \\
&+ s_{m-1}u_{l} \left( \prod_{k=0}^{m-1} \prod_{i=m-k-1}^{m-1} \prod_{j=n-l}^{m-1} \left( -\frac{s_j}{r_i} \right) \left( -\frac{u_j}{t_j} \right) y_{m-k-1,n-l} \right) \\
&+ r_{m}u_{n-1} \left( \prod_{k=0}^{m-1} \prod_{i=0}^{m-k-1} \prod_{j=n-l}^{k} \left( -\frac{s_j}{r_i} \right) \left( -\frac{u_j}{t_j} \right) y_{m-k-1,n-l-1} \right) \\
&+ r_{m}u_{n} \left( \prod_{k=0}^{m-1} \prod_{i=m-k-1}^{m-1} \prod_{j=n-l}^{m-1} \left( -\frac{s_j}{r_i} \right) \left( -\frac{u_j}{t_j} \right) y_{m-k-1,n-l} \right) \\
&= \sum_{k=0}^{m-1} \sum_{l=0}^{n-1} \sum_{i=0}^{m-k-1} \sum_{j=n-l}^{m-1} \left( -\frac{s_j}{r_i} \right) \left( -\frac{u_j}{t_j} \right) y_{m-k-1,n-l-1} \\
&+ \sum_{k=0}^{m-1} \sum_{l=0}^{n-1} \sum_{i=m-k-1}^{m-1} \sum_{j=n-l}^{m-1} \left( -\frac{s_j}{r_i} \right) \left( -\frac{u_j}{t_j} \right) y_{m-k-1,n-l} \\
&+ \sum_{k=0}^{m-1} \sum_{l=0}^{n-1} \sum_{i=0}^{m-k-1} \sum_{j=n-l}^{m-1} \left( -\frac{s_j}{r_i} \right) \left( -\frac{u_j}{t_j} \right) y_{m-k-1,n-l-1} \\
&+ \sum_{k=0}^{m-1} \sum_{l=0}^{n-1} \sum_{i=m-k-1}^{m-1} \sum_{j=n-l}^{m-1} \left( -\frac{s_j}{r_i} \right) \left( -\frac{u_j}{t_j} \right) y_{m-k-1,n-l} \\
&= y_{mn}
\end{align*}
\]

for all \( m, n \in \mathbb{N} \). This equality leads us to the fact that

\[
p - \lim_{q, q' \to \infty, n > 0} \sup \frac{1}{(q + 1)(q' + 1)} \sum_{k=m}^{m+q} \sum_{l=n}^{m+q'} (\tilde{B}x)_{kl} | = p - \lim_{q, q' \to \infty, n > 0} \sup \frac{1}{(q + 1)(q' + 1)} \sum_{k=m}^{m+q} \sum_{l=n}^{m+q'} y_{kl} |.
\]

This shows us that \( x = (x_{mn}) \in \tilde{B}(\mathcal{E}_f) \) since \( y = (y_{mn}) \) lies in \( \mathcal{E}_f \). Thus, \( T \) is surjective. Therefore, \( T \) is a linear bijection between the spaces \( \tilde{B}(\mathcal{E}_f) \) and \( \mathcal{E}_f \). It completes the proof. \( \square \)

**Theorem 0.2.** The following inclusions strictly hold.

(i) \( \mathcal{E}_f \subset \tilde{B}(\mathcal{E}_f) \).

(ii) \( \mathcal{E}_f \subset \tilde{B}(\mathcal{E}_f) \).

(iii) \( [\mathcal{E}_f] \subset \tilde{B}(\mathcal{E}_f) \).
(iv) \([\mathscr{C}_f] \subset \widetilde{B}[\mathscr{C}_f]\).

(v) \(\widetilde{B}[\mathscr{C}_f] \subset \widetilde{B}(\mathscr{C}_f)\).

(vi) \(\widetilde{B}[\mathscr{C}_f] \subset \widetilde{B}(\mathscr{C}_f)\).

Proof. (i - ii) : Suppose that \(x = (x_{kl})\) be a double sequence in the set \(\mathscr{C}_f\) such that

\[
p - \lim_{q, q' \to 0} \sup_{m, n > 0} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} (Bx)_{kl} \right|
\]

exists uniformly in \(m, n \in \mathbb{N}\). We need to show that the double sequence \(x = (x_{kl})\) is also in \(\widetilde{B}(\mathscr{C}_f)\), i.e., \(Bx \in \mathscr{C}_f\). Thus, we have

\[
\left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} (Bx)_{kl} \right| \leq \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} (s_{k-1}u_{l, j}x_{k-1, l-1} + s_{k-1}t_{l}x_{k-1, l-1} + r_{k}u_{l, j}x_{k, l-1} + r_{k}t_{l}x_{k, l-1}) \right|
\]

Hence, if we pass \(p\) - limit by letting \(q, q' \to \infty\), then

\[
p - \lim_{q, q' \to 0} \sup_{m, n > 0} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} (Bx)_{kl} \right| \leq \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} (s_{k-1}u_{l, j}x_{k-1, l-1} + s_{k-1}t_{l}x_{k-1, l-1} + r_{k}u_{l, j}x_{k, l-1} + r_{k}t_{l}x_{k, l-1}) \right|
\]

for all \(k, l \in \mathbb{N}\). Clearly \(x = (x_{kl})\) is not in \(\mathscr{C}_f\). Furthermore,

\[
(Bx)_{kl} = \frac{1}{r_{k}t_{l}} \prod_{i=0}^{k-1} \prod_{j=0}^{l-1} \left( -s_{i} \right) \left( -u_{j} \right) \left( \frac{-u_{j}}{t_{j}} \right)
\]

which means that the inclusion \(\mathscr{C}_f \subset \widetilde{B}(\mathscr{C}_f)\) holds.

Now, we must show that the set \(\widetilde{B}(\mathscr{C}_f) \setminus \mathscr{C}_f\) is not empty. Let us define \(x = (x_{kl})\) by

\[
x_{kl} = \frac{1}{r_{k}t_{l}} \prod_{i=0}^{k-1} \prod_{j=0}^{l-1} \left( -s_{i} \right) \left( -u_{j} \right) \left( \frac{-u_{j}}{t_{j}} \right)
\]

for all \(k, l \in \mathbb{N}\). Clearly \(x = (x_{kl})\) is not in \(\mathscr{C}_f\). Furthermore,

\[
(Bx)_{kl} = \frac{1}{r_{k}t_{l}} \prod_{i=0}^{k-1} \prod_{j=0}^{l-1} \left( -s_{i} \right) \left( -u_{j} \right) \left( \frac{-u_{j}}{t_{j}} \right)
\]

that is, \(Bx = 0\) for every \(k, l \in \mathbb{N}\), which is in the space \(\mathscr{C}_f\), i.e., \(x \in \widetilde{B}(\mathscr{C}_f) \setminus \mathscr{C}_f\). This gives us that the inclusion \(\mathscr{C}_f \subset \widetilde{B}(\mathscr{C}_f)\) is strict.

(iii - iv) : Similarly, let us assume that \(x = (x_{kl})\) be a double sequence in the space \(\mathscr{C}_f\) such that

\[
p - \lim_{q, q' \to 0} \sup_{m, n > 0} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} |x_{kl}| \right|
\]
exists uniformly in \(m,n \in \mathbb{N}\). We must show here that the double sequence \(x = (x_{kl})\) is also in \(\tilde{B}[\mathcal{C}_f]\), i.e., \(\tilde{B}x \in [\mathcal{C}_f]\). Thus, we have

\[
\frac{1}{(q+1)(q'+1)} \sum_{k,m,l=1}^{m+q, n+q'} |(Bx)_{kl}| 
= \frac{1}{(q+1)(q'+1)} \sum_{k,m,l=1}^{m+q, n+q'} |s_{k-1}u_{l-1}x_{k-1,l-1} + s_{k-1}u_{l-1}x_{k,l-1} + r_ku_{l-1}x_{k,l}| 
\leq \frac{1}{(q+1)(q'+1)} \sum_{k,m,l=1}^{m+q, n+q'} |s_{k-1}u_{l-1}x_{k-1,l-1} + s_{k-1}u_{l-1}x_{k,l-1} + r_ku_{l-1}x_{k,l}| 
+ \frac{1}{(q+1)(q'+1)} \sum_{k,m,l=1}^{m+q, n+q'} |s_{k-1}u_{l-1}x_{k,l-1} + s_{k-1}u_{l-1}x_{k,l-1} + r_ku_{l-1}x_{k,l}| 
\leq \frac{1}{(q+1)(q'+1)} \sum_{k,m,l=1}^{m+q, n+q'} |x_{k-1,l-1}| + \frac{1}{(q+1)(q'+1)} \sum_{k,m,l=1}^{m+q, n+q'} |x_{k,l-1}| 
+ \frac{1}{(q+1)(q'+1)} \sum_{k,m,l=1}^{m+q, n+q'} |x_{k,l-1}| + \frac{1}{(q+1)(q'+1)} \sum_{k,m,l=1}^{m+q, n+q'} |x_{k,l}| .
\]

Hence, if we pass \(p\)-limit by letting \(q,q' \to \infty\), then

\[
p - \lim_{q,q' \to \infty} \sup_{m,n \geq 0} \frac{1}{(q+1)(q'+1)} \sum_{k,m,l=1}^{m+q, n+q'} |(Bx)_{kl}| 
= \frac{1}{(q+1)(q'+1)} \sum_{k,m,l=1}^{m+q, n+q'} |(Bx)_{kl}| 
\leq \frac{1}{(q+1)(q'+1)} \sum_{k,m,l=1}^{m+q, n+q'} |x_{kl}| .
\]

This gives us that the inclusion \([\mathcal{C}_f] \subset \tilde{B}[\mathcal{C}_f]\) holds.

Now, to show that the set \(\tilde{B}[\mathcal{C}_f] \setminus [\mathcal{C}_f]\) is not empty, we may consider the double sequence \(x = (x_{kl})\) defined as (7) such that \(\tilde{B}x = 0\) which is also in the space \([\mathcal{C}_f]\), i.e., \(x \in \tilde{B}[\mathcal{C}_f] \setminus [\mathcal{C}_f]\). This gives us that the inclusion \([\mathcal{C}_f] \subset \tilde{B}[\mathcal{C}_f]\) is strict.

\((v \rightarrow vi)\) : The proofs can be proved easily since \([\mathcal{C}_f] \subset \mathcal{C}_f\) and \([\mathcal{C}_f] \subset \mathcal{C}_f\) and the double sequence \(x = (x_{kl})\) defined as in (7) shows that the sets \(\mathcal{C}_f \setminus [\mathcal{C}_f]\) and \(\mathcal{C}_f \setminus [\mathcal{C}_f]\) are not empty. So, we omit the details.

\[\square\]

**Theorem 0.3.** The following statements hold.

(a) If \(\sup_{m,n} \mu_{m,n} < 1\) for all \(m \in \mathbb{N}\) and \(\sup_{m,n} \mu_{m,n} < 1\) for all \(n \in \mathbb{N}\). The spaces \(\mathcal{M}_u\) and \(\mu\) do not contain each other where \(\mu = \{\tilde{B}[\mathcal{C}_f], \tilde{B}[\mathcal{C}_f]\}\).

(b) If \(\sup_{m,n} \mu_{m,n} < 1\) for all \(m \in \mathbb{N}\) and \(\sup_{m,n} \mu_{m,n} < 1\) for all \(n \in \mathbb{N}\). Then, the following inclusions \(\tilde{B}[\mathcal{C}_f] \subset \mathcal{M}_u\) and \(\tilde{B}[\mathcal{C}_f] \subset \mathcal{M}_u\) strictly hold.

**Proof.** (a) : To prove this claim, we must prove that the sets \(\tilde{B}[\mathcal{C}_f] \setminus \mathcal{M}_u\), \(\mathcal{M}_u \setminus \tilde{B}[\mathcal{C}_f]\) and \(\tilde{B}[\mathcal{C}_f] \cap \mathcal{M}_u\) are not empty. Let us define a double sequence \(x = (x_{kl})\) by

\[x_{kl} = \frac{k(\ell-1)^l}{r_k\ell l} \text{ for all } k,l \in \mathbb{N}.
\]

Thus, \(\tilde{B}x \in [\mathcal{C}_f]\) since \(\sup_{m,n} \mu_{m,n} < 1\) for all \(m \in \mathbb{N}\) and \(\sup_{m,n} \mu_{m,n} < 1\) for all \(n \in \mathbb{N}\), but clearly not in the space \(\mathcal{M}_u\). The set \(\tilde{B}[\mathcal{C}_f] \setminus \mathcal{M}_u\) is not empty. Moreover, if we define a double sequence \(x^{(1)} = (x_{kl}^{(1)})\) by \(x^{(1)} = e\) then, it is clear that \(x^{(1)} = (x_{kl}^{(1)})\) is in \(\mathcal{M}_u\).

Now, if we define a double sequence \(x^{(2)}\) which is in the space \(\mathcal{M}_u\) by

\[x_{kl}^{(2)} = \begin{cases} \frac{1}{r_k\ell l}, & \text{if both } k \text{ and } l \text{ are even} \\ 0, & \text{otherwise}. \end{cases}
\]
Then, we have the $\tilde{B}$–transform of $x = (x_{kl}^{(2)})$ as

$$
(\tilde{B}x)_{kl} = \begin{cases} 
\frac{r_{2k-1}}{r_{2l-1}}, & \text{if } k \text{ is odd } l \text{ is odd}, \\
\frac{r_{2k-2}}{r_{2l-1}}, & \text{if } k \text{ is odd } l \text{ is even}, \\
\frac{r_{2k-1}}{r_{2l}}, & \text{if } k \text{ is even } l \text{ is odd}, \\
\frac{r_{2k-2}}{r_{2l}}, & \text{if } k \text{ is even } l \text{ is even}.
\end{cases}
$$

Thus, one can obtain since $\frac{\sup_m s_n}{\inf_m r_n} < 1$ for all $m \in \mathbb{N}$ and $\frac{\sup_n u_n}{\inf_n t_n} < 1$ for all $n \in \mathbb{N}$ that

$$
p - \lim_{q,q' \to \infty} m,n \geq 0 \sup \left( q + 1 \right) \left( q' + 1 \right) \sum_{k=m}^{m+q} \sum_{l=n}^{n+q} \left| (\tilde{B}x)_{kl} \right| = 1
$$

uniformly in $m,n \in \mathbb{N}$. We read from the last approach that $X^{(2)} \in \mathcal{M}_u \setminus \tilde{B}(\mathcal{E}_f)$. This is what we claimed.

(b) : First, we need to show that the inclusions $\tilde{B}(\mathcal{E}_f) \subset \mathcal{M}_u$ and $\tilde{B}(\mathcal{E}_f) \subset \mathcal{M}_u$ hold. Let us take any double sequence $x = (x_{kl}) \in \tilde{B}(\mathcal{E}_f)$. Then $y = \tilde{B}x \in \mathcal{E}_f \subset \mathcal{M}_u$. Since $B^{-1}(\tilde{r},\tilde{s},\tilde{t},\tilde{u})$ satisfy the conditions of (Tuğ, 2018, Theorem 4.10), it belongs to class $\mathcal{F}_{\mathcal{M}_u}$ and $x = \tilde{B}^{-1}y \in \mathcal{M}_u$ holds. Therefore, the inclusion $\tilde{B}(\mathcal{E}_f) \subset \mathcal{M}_u$ holds.

Now, for the converse, let the following double sequence $x = (x_{kl})$ be defined as

$$
x_{kl} = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\end{pmatrix}, \quad (9)
$$

i.e., in each row, there is one 1, then two 0s, then four 1s, then eight 0s, then sixteen 1s, and goes on with respect to this harmony. Thus, clearly the double sequence $x = (x_{kl}) \in \mathcal{M}_u \setminus \tilde{B}(\mathcal{E}_f)$. \qed

3. The $\alpha-, \beta(bp)-$ and $\gamma-$ duals of the spaces $\tilde{B}(\mathcal{E}_f)$ and $\tilde{B}(\mathcal{E}_f)$

In this section, first we calculate the $\alpha-$ dual of the spaces $\tilde{B}(\mathcal{E}_f)$ and $\tilde{B}(\mathcal{E}_f)$. Then we state some known lemmas concerning the matrix classes $\mathcal{M}_u$ and $\mathcal{M}_u$ which we consider them to calculate the $\beta(bp)-$ and $\gamma-$ duals of the spaces $\tilde{B}(\mathcal{E}_f)$ and $\tilde{B}(\mathcal{E}_f)$.

**Theorem 0.4.** Let $\frac{\sup_m s_n}{\inf_m r_n} < 1$ for all $m \in \mathbb{N}$ and $\frac{\sup_n u_n}{\inf_n t_n} < 1$ for all $n \in \mathbb{N}$. Then, the $\alpha-$ dual of the spaces $\tilde{B}(\mathcal{E}_f)$ and $\tilde{B}(\mathcal{E}_f)$ is the space $\mathcal{L}_u$.

**Proof.** Since the proofs are similar to each other, we only prove here $\tilde{B}(\mathcal{E}_f) = \mathcal{L}_u$ and leave the other assumption to the reader. To prove our claim $\tilde{B}(\mathcal{E}_f) = \mathcal{L}_u$, we must show the existence of the inclusions $\mathcal{L}_u \subset \tilde{B}(\mathcal{E}_f)$ and $\tilde{B}(\mathcal{E}_f) \subset \mathcal{L}_u$.

For the first inclusion, suppose a sequence $a = (a_{mn}) \in \mathcal{L}_u$ and $x = (x_{mn}) \in \tilde{B}(\mathcal{E}_f)$. Then, there exists a double sequence $y = (y_{mn}) \in \mathcal{E}_f$ with the relation (3) and (5) such that

$$
p - \lim_{q,q' \to \infty} m,n \geq 0 \sup \left( q + 1 \right) \left( q' + 1 \right) \sum_{k=m}^{m+q} \sum_{l=n}^{n+q} \left| y_{kl} \right| = 1
$$

exists. Moreover, the inclusion $\mathcal{E}_f \subset \mathcal{M}_u$ holds, says, $\sup_{m,n \in \mathbb{N}} |y_{mn}| \leq K$ where $K \in \mathbb{R}^+$. Since $\frac{\sup_m s_n}{\inf_m r_n} < 1$ for all $m \in \mathbb{N}$ and $\frac{\sup_n u_n}{\inf_n t_n} < 1$ for all $n \in \mathbb{N}$, we have

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\[
\sum_{m,n} |a_{mn}x_{mn}| = \sum_{m,n} |a_{mn}| \left| \frac{1}{\prod_{k,l=0}^{n-1} \prod_{i=m-k}^{m-1} \prod_{j=n-l}^{n-1} \left( \frac{-s_i}{r_i} \right) \left( \frac{-u_j}{t_j} \right) } \right| y_{m-k,n-l} \\
\leq \frac{1}{||f||} \sum_{m,n} |a_{mn}| \sum_{k,l=0}^{n-1} \prod_{i=m-k}^{m-1} \prod_{j=n-l}^{n-1} \left| \frac{-s_i}{r_i} \right| \left| \frac{-u_j}{t_j} \right| y_{m-k,n-l} \\
\leq K \frac{1}{||f||} \sum_{m,n} |a_{mn}| \left( \sum_{k,l=0}^{n-1} \sup_{i,m} s_i \inf_{r_m} r_m \right) \left( \sum_{k,l=0}^{n-1} \inf_{i,n} u_l \right) y_{m-k,n-l} \\
= \frac{K}{||f||} \left( \sum_{m,n} |a_{mn}| \right) \left( \sum_{m,n} \left( 1 - \frac{1}{\sup_{i,m} s_i \inf_{r_m} r_m} \right) \right) \left( \sum_{m,n} \left( 1 - \frac{1}{\inf_{i,n} u_l} \right) \right) < \infty.
\]

Thus, \( a = (a_{mn}) \in \left\{ \tilde{B}(B(f)) \right\}^\alpha \) and we can say that the fact that the inclusion \( \mathcal{L}_u \subset \left\{ \tilde{B}(B(f)) \right\}^\alpha \) holds.

Now, to prove the second inclusion \( \left\{ \tilde{B}(B(f)) \right\}^\alpha \subset \mathcal{L}_u \), we use converse of this assumption. Let suppose that there exists a sequence \( (a_{mn}) \in \left\{ \tilde{B}(B(f)) \right\}^\alpha \setminus \mathcal{L}_u \) such that \( \sum_{m,n} |a_{mn}x_{mn}| < \infty \) for all \( x = (x_{mn}) \in \tilde{B}(B(f)) \). Let the double sequence \( x = (x_{kl}) \) be defined as \( x = (x_{mn}) = \{(-1)^{m+n}\} \) which is in \( \tilde{B}(B(f)) \) such that
\[
\sum_{m,n} |a_{mn}x_{mn}| = \sum_{m,n} |a_{mn}| = \infty.
\]

Thus this is a contradiction. Therefore, \( (a_{mn}) \) must belong to the space \( \mathcal{L}_u \). It completes the proof. \( \square \)

**Lemma 0.5.** (Moricz & Rhoades, 1988, Theorem 1., p.285) The following statements hold:

(a) A four-dimensional matrix \( A = (a_{mnkl}) \in (\mathcal{C}_f : \mathcal{C}_{bp}) \) if and only if the following conditions hold:
\[
\sup_{m,n \in \mathbb{N}} \sum_{k,l} |a_{mnkl}| < \infty \tag{10}
\]
\[
\exists a_{kl} \in \mathbb{C}, b_{p} - \lim_{m,n \rightarrow \infty} a_{mnkl} = a_{kl} \text{ for all } k,l \in \mathbb{N}, \tag{11}
\]
\[
\exists u \in \mathbb{C}, b_{p} - \lim_{m,n \rightarrow \infty} a_{mnkl} = u, \tag{12}
\]
\[
\exists k_{0} \in \mathbb{N}, b_{p} - \lim_{m,n \rightarrow \infty} \sum_{l} |a_{mn,k_{0}l} - a_{k_{0}l}| = 0 \text{ for all } l \in \mathbb{N}, \tag{13}
\]
\[
\exists l_{0} \in \mathbb{N}, b_{p} - \lim_{m,n \rightarrow \infty} \sum_{k} |a_{mn,k_{0}l} - a_{k_{0}l}| = 0 \text{ for all } k \in \mathbb{N}, \tag{14}
\]
\[
b_{p} - \lim_{m,n \rightarrow \infty} \sum_{k} \sum_{l} |\Delta_{0} a_{mnkl}| = 0, \tag{15}
\]
\[
b_{p} - \lim_{m,n \rightarrow \infty} \sum_{k} \sum_{l} |\Delta_{10} a_{mnkl}| = 0, \tag{16}
\]
where
\[
\Delta_{10} a_{mnkl} = a_{mnkl} - a_{mn,k+1,l}, \quad \Delta_{01} a_{mnkl} = a_{mnkl} - a_{mn,k,l+1}. \tag{17}
\]

(b) A four-dimensional matrix \( A = (a_{mnkl}) \) is strongly regular, i.e., \( A \in (\mathcal{C}_f : \mathcal{C}_{bp})_{\text{reg}} \) if and only if the conditions (10)-(16) hold with \( a_{kl} = 0 \) for all \( k,l \in \mathbb{N} \) and \( u = 1 \).

where \( \Delta_{10} a_{mnkl} = a_{mnkl} - a_{m,n,k+1,l} \) and \( \Delta_{01} a_{mnkl} = a_{mnkl} - a_{m,n,k,l+1} \), \( (m,n,k,l = 0,1,2,...) \).
Lemma 0.6. (Başarir, 1995, Theorem 1., p.179) A four-dimensional matrix \( A = (a_{mnkl}) \in ([\mathscr{C}_f] : \mathscr{C}_p) \) if and only if \( A \) is bounded regular, i.e., \( A = (a_{mnkl}) \in ([\mathscr{C}_f] : \mathscr{C}_p) \) (see (Robison, 1926, Theorem 1., p.53)), that is the conditions in (11)-(14) hold with \( a_{kl} = 0 \) for all \( k,l \in \mathbb{N} \) and \( u = 1 \), and satisfy the following two conditions:

\[
\begin{align*}
bp &= \lim_{m,n \to \infty} \sum_{k,l \in E} |\Delta_{10} a_{mnkl}| = 0, \\
bp &= \lim_{m,n \to \infty} \sum_{k,l \in E} |\Delta_{01} a_{mnkl}| = 0.
\end{align*}
\]

for each set \( E \) which is uniformly zero density.

Lemma 0.7. (Tuğ, 2018, Theorem 4.10, p.14) A four-dimensional matrix \( A = (a_{mnkl}) \in ([\mathscr{C}_f] : \mathscr{M}_u) \) if and only if \( A_{mn} \in \{ \mathscr{C}_f \}^{\beta_{(o)}} \) and condition (10) hold.

Lemma 0.8. (Tuğ, 2021, Corollary 3.4, p.13) A four-dimensional matrix \( A = (a_{mnkl}) \in ([\mathscr{C}_f] : \mathscr{M}_u) \) if and only if the \( A_{mn} \in \{ \mathscr{C}_f \}^{\beta_{(o)}} \) for all \( m,n \in \mathbb{N} \) and (10) holds.

Now let us define the sets \( \tilde{b}_k \) where \( k \in \{1, 2, \ldots, 7\} \), as follows:

\[ \tilde{b}_1 = \left\{ a = (a_{kl}) \in \Omega : \sup_{m,n \in \mathbb{N}} \sum_{k,l \in \mathbb{N}} \sum_{i,j=k,l}^{m,n} \left( \prod_{r=1}^{i-1} \prod_{t=1}^{j-1} \left( \frac{-s \rho}{r \pi} \right) \right) a_{ij} < \infty \right\}, \]

\[ \tilde{b}_2 = \left\{ a = (a_{kl}) \in \Omega : \exists \beta_{kl} \subseteq \mathbb{C} \ni \theta - \lim_{m,n \to \infty} \sum_{i,j=k,l}^{m,n} \left( \prod_{r=1}^{i-1} \prod_{t=1}^{j-1} \left( \frac{-s \rho}{r \pi} \right) \right) a_{ij} = \beta_{kl} \right\}, \]

\[ \tilde{b}_3 = \left\{ a = (a_{kl}) \in \Omega : \exists \theta \subseteq \mathbb{C} \ni \theta - \lim_{m,n \to \infty} \sum_{i,j=k,l}^{m,n} \left( \prod_{r=1}^{i-1} \prod_{t=1}^{j-1} \left( \frac{-s \rho}{r \pi} \right) \right) a_{ij} = \theta \right\}, \]

\[ \tilde{b}_4 = \left\{ a = (a_{kl}) \in \Omega : \exists \alpha \subseteq \mathbb{N} \ni \alpha - \lim_{m,n \to \infty} \sum_{i,j=k,l}^{m,n} \left( \prod_{r=1}^{i-1} \prod_{t=1}^{j-1} \left( \frac{-s \rho}{r \pi} \right) \right) a_{ij} = \alpha \right\}, \]

\[ \tilde{b}_5 = \left\{ a = (a_{kl}) \in \Omega : \exists \gamma \subseteq \mathbb{N} \ni \gamma - \lim_{m,n \to \infty} \sum_{i,j=k,l}^{m,n} \left( \prod_{r=1}^{i-1} \prod_{t=1}^{j-1} \left( \frac{-s \rho}{r \pi} \right) \right) a_{ij} = \gamma \right\}, \]

\[ \tilde{b}_6 = \left\{ a = (a_{kl}) \in \Omega : \exists \delta \subseteq \mathbb{N} \ni \delta - \lim_{m,n \to \infty} \sum_{i,j=k,l}^{m,n} \left( \prod_{r=1}^{i-1} \prod_{t=1}^{j-1} \left( \frac{-s \rho}{r \pi} \right) \right) a_{ij} = \delta \right\}, \]

Theorem 0.9. The \( \beta_{(bp)} = \text{dual of } \{ \mu \}^{\beta_{(bp)}} \) of the space \( \mu = \bigcap_{i=1}^{7} \tilde{b}_i \), i.e., \( \{ \mu \}^{\beta_{(bp)}} = \bigcap_{i=1}^{7} \tilde{b}_i \), where \( \mu = \{ \tilde{B}(\mathscr{C}_f), \tilde{B}(\mathscr{C}_p) \} \).

**Proof.** Suppose that \( a = (a_{mn}) \in \Omega \) and \( x = (x_{mn}) \in \tilde{B}(\mathscr{C}_f) \). Then, we have \( y = \tilde{B}x \in \mathscr{C}_f \), \( m,n \)-th partial sum of the series \( \sum_{k,l} a_{kl}x_{kl} \) is given by the equality (26) which was defined by Tuğ at al (Tuğ et al., 2020, Lemma 1., p.11) and the four-dimensional matrix \( \tilde{D} = (\tilde{a}_{mnkl}) \) which was also defined by Tuğ at al (Tuğ et al., 2020, p. 11) as

\[ \tilde{a}_{mnkl} = \begin{cases} \sum_{i,j=k,l}^{m,n} \frac{1}{r t} \prod_{r=1}^{i-1} \prod_{t=1}^{j-1} \left( \frac{-s \rho}{r \pi} \right) a_{ij}, & 0 \leq k \leq m, 0 \leq l \leq n; \\ 0, & \text{elsewhere} \end{cases} \]
for all \(m,n,k,l \in \mathbb{N}\). Then, since it is given in the hypothesis, one can obtain that \(ax \in \mathcal{CS}_{bp}\) whenever \(x = (x_{mn}) \in \tilde{B}[\mathcal{E}_f]\) if and only if \(\tilde{D}y \in \mathcal{CS}_{bp}\) whenever \(y = (y_{mn}) \in \mathcal{E}_f\). This says that \(a = (a_{mn}) \in \{\tilde{B}(\mathcal{E}_f)\}^{-\beta(bp)}\) if and only if \(\tilde{D} \in (\mathcal{E}_f : \mathcal{CS}_{bp})\). Thus, we can say that the conditions of Lemma 0.5(a) holds with \(\tilde{d}_{mnkl}\) instead of \(a_{mn}\), i.e., this is the set \(\bigcap_{i=1}^{7} \tilde{d}_i\). This completes the proof.

**Theorem 0.10.** The \(\gamma\)-dual of the spaces \(\tilde{B}(\mathcal{E}_f)\) and \(\tilde{B}[\mathcal{E}_f]\) is the set \(\tilde{d}_1 \cap CS_{\phi}\), i.e., \(\{\tilde{B}(\mathcal{E}_f)\}^{\gamma} = \{\tilde{B}[\mathcal{E}_f]\}^{\gamma} = \bigcap_{i=1}^{7} \tilde{d}_i \cap CS_{\phi}\)

**Proof.** Suppose that \(a = (a_{mn}) \in \Omega\) and \(x = (x_{mn}) \in \tilde{B}[\mathcal{E}_f]\). We need to show that the \((m,n)^{th}\) partial sum of the series \(\sum_{kl} a_{kl}x_{kl}\) is in the space \(\mathcal{B}[\mathcal{E}_f]\) for these sequences \(a = (a_{mn}) \in \Omega\) and \(x = (x_{mn}) \in \tilde{B}[\mathcal{E}_f]\) where \(y = \tilde{B}x \in [\mathcal{E}_f]\). By considering the similar way used in proving Theorem 0.9, we can summarize the rest of the proof as follows; we can say that \(ax \in \mathcal{B}[\mathcal{E}_f]\) whenever \(x = (x_{mn}) \in \tilde{B}[\mathcal{E}_f]\) if and only if \(\tilde{D}y \in M_a\) whenever \(y = (y_{mn}) \in [\mathcal{E}_f]\), where the matrix \(\tilde{D} = (\tilde{d}_{mnkl})\) was defined by (20). This means that the conditions of Lemma 0.8 hold with the matrix \(\tilde{D} = (\tilde{d}_{mnkl})\) instead of the matrix \(A = (a_{mn})\). That is, \(D_{mn} \in [\mathcal{E}_f]^{\beta(\phi)}\) for each fixed \(m,n \in \mathbb{N}\) and

\[
\sup_{m,n \in \mathbb{N}} \sum_{k,l} \frac{1}{i} \prod_{t,j=1}^{i-1} \prod_{r,s=1}^{j-1} \left( \frac{s}{r} \right)^{m,n} a_{ij} \leq \infty.
\]

Thus, the \(\gamma\)-dual \(\{\tilde{B}[\mathcal{E}_f]\}^{\gamma}\) of the space \(\tilde{B}[\mathcal{E}_f]\) is the set \(\bigcap_{i=1}^{7} \tilde{d}_i \cup CS_{\phi}\). This completes the proof. \(\square\)

4. Matrix Transformations on the New Sequence Spaces \(\tilde{B}(\mathcal{E}_f)\) and \(\tilde{B}[\mathcal{E}_f]\)

In this section, first we summarize the literature concerning the matrix transformations from and into the sequence spaces \(\mathcal{E}_f\) and \([\mathcal{E}_f]\). Then we state some corollaries, without their proofs, which include characterization of some new four-dimensional matrix classes. We conclude the section with some significant results after stating the matrix classes \((\tilde{B}(\mu) : \lambda)\) and \((\lambda : \tilde{B}(\mu))\) which characterized by Tuğ et al. (2020) in general form.

**Lemma 0.11.** (Zeltser, Mursaleen, & Mohiuddine, 2009, Theorem 3.1., p. 5) The following statements hold:

(a) A four-dimensional matrix \(A = (a_{mnkl})\) is almost \(\mathcal{CS}_{bp}\)-conservative, i.e., \(A \in (\mathcal{CS}_{bp} : \mathcal{E}_f)\) if and only
if the condition in (10), and the following conditions hold

\[ \exists a_{ij} \in \mathbb{C} \ni b - \lim_{q,q' \to \infty} a(i,j,q,q',m,n) = a_{ij}, \quad (20) \]
uniformlly in \( m,n \in \mathbb{N} \) for each \( i,j \in \mathbb{N} \)

\[ \exists u \in \mathbb{C} \ni b - \lim_{q,q' \to \infty} \sum_{i,j} a(i,j,q,q',m,n) = u, \quad (21) \]
uniformlly in \( m,n \in \mathbb{N} \)

\[ \exists a_{ij} \in \mathbb{C} \ni b - \lim_{q,q' \to \infty} \sum_{i} |a(i,j,q,q',m,n) - a_{ij}| = 0, \quad (22) \]
uniformlly in \( m,n \in \mathbb{N} \) for each \( i \in \mathbb{N} \)

\[ \exists a_{ij} \in \mathbb{C} \ni b - \lim_{q,q' \to \infty} \sum_{j} |a(i,j,q,q',m,n) - a_{ij}| = 0, \quad (23) \]
uniformlly in \( m,n \in \mathbb{N} \) for each \( j \in \mathbb{N} \)

where \( a(i,j,q,q',m,n) = \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl} / [(q+1)(q'+1)] \). In this case, \( a = (a_{ij}) \in \mathcal{L}_u \) and

\[ f_2 - \lim Ax = \sum_{i,j} a_{ij}x_{ij} + \left( u - \sum_{i,j} a_{ij} \right) b - \lim_{i,j \to \infty} x_{ij}, \]
that is,

\[ b - \sum_{q,q' \to \infty} \sum_{i,j} a(i,j,q,q',m,n)x_{ij} = \sum_{i,j} a_{ij}x_{ij} + \left( u - \sum_{i,j} a_{ij} \right) b - \lim_{i,j \to \infty} x_{ij}, \]
uniformlly in \( m,n \in \mathbb{N} \).

(b) A four-dimensional matrix \( A = (a_{mnkl}) \) is almost \( \mathcal{C}_{bp} \)-regular, i.e., \( A \in (\mathcal{C}_{bp} : \mathcal{C}_{f})_{\text{reg}} \) if and only if the conditions (10), (20)-(23) hold with \( a_{ij} = 0 \) for all \( i,j \in \mathbb{N} \) and \( u = 1 \)

Lemma 0.12. (Zeltser et al., 2009, Theorem 3.2., p.9) The following statements hold:

(a) A four-dimensional matrix \( A = (a_{mnkl}) \) is almost \( \mathcal{C}_{r} \)-conservative, i.e., \( A \in (\mathcal{C}_{r} : \mathcal{C}_{f}) \) if and only if the conditions in (10), (20) and (21), and the following conditions hold

\[ \exists j_0 \in \mathbb{N} \ni b - \lim_{q,q' \to \infty} \sum_{i} a(i,j_0,q,q',m,n) = u_{j_0}, \quad (24) \]
uniformlly in \( m,n \in \mathbb{N} \),

\[ \exists l_0 \in \mathbb{N} \ni b - \lim_{q,q' \to \infty} \sum_{j} a(i_0,j,q,q',m,n) = v_{l_0}, \quad (25) \]
uniformlly in \( m,n \in \mathbb{N} \)

where \( a(i,j,q,q',m,n) \) is defined as in the Lemma 0.11. In this case, \( a = (a_{ij}) \in \mathcal{L}_u; (u_j), (v_i) \in \ell_1 \) and

\[ f_2 - \lim Ax = \sum_{i,j} a_{ij}x_{ij} + \sum_{i} \left( v_{i} - \sum_{j} a_{ij} \right) x_{i} + \sum_{j} \left( u_{j} - \sum_{i} a_{ij} \right) x_{j} \]

\[ + \left( u + \sum_{i,j} a_{ij} - \sum_{i} v_{i} - \sum_{j} u_{j} \right) r - \lim_{i,j \to \infty} x_{ij}. \]
(b) A four-dimensional matrix $A = (a_{mnkl})$ is almost $\mathcal{C}_r$-regular, i.e., $A \in (\mathcal{C}_r : \mathcal{C}_f)_{\text{reg}}$ if and only if the conditions in (10), (20), (21), (25) and (25) hold with $a_{ij} = u_j = v_i = 0$ for all $i, j \in \mathbb{N}$ and $u = 1$.

**Lemma 0.13.** (Zeltser et al., 2009, Theorem 3.3., p.11) The following statements hold:

(a) A four-dimensional matrix $A = (a_{mnkl})$ is almost $\mathcal{C}_p$-conservative, i.e., $A \in (\mathcal{C}_p : \mathcal{C}_f)_{\text{reg}}$ if and only if the conditions in (10), (20) and (21) hold, and

\[
\forall k \in \mathbb{N}, \exists K \in \mathbb{N} \ni a_{mnkl} = 0 \text{ for } l > K, \ (m, n \in \mathbb{N}), \tag{26}
\]

\[
\forall l \in \mathbb{N}, \exists L \in \mathbb{N} \ni a_{mnkl} = 0 \text{ for } k > L, \ (m, n \in \mathbb{N}). \tag{27}
\]

In this case $a = (a_{ij}) \in \mathcal{L}_u, (a_{ij})_{i \in \mathbb{N}, j \in \mathbb{N}} \in \varphi$ where $\varphi$ denotes the space of all finitely non-zero sequences and

\[
f_2 - \lim_{x \to \infty} A x = \sum_{i,j} a_{ij} x_{ij} + \left( u - \sum_{i,j} a_{ij} \right) p - \lim_{x \to \infty} x_{ij}.
\]

(b) A four-dimensional matrix $A = (a_{mnkl})$ is almost $\mathcal{C}_p$-regular, i.e., $A \in (\mathcal{C}_p : \mathcal{C}_f)_{\text{reg}}$ if and only if the conditions in (10), (20), (21), (26) and (27) hold with $a_{ij} = 0$ for all $i, j \in \mathbb{N}$ and $u = 1$.

**Lemma 0.14.** (Mursaleen, 2004, Theorem 2.2., p.527) A four-dimensional matrix $A = (a_{mnkl})$ is almost strongly regular, i.e., $A \in (\mathcal{C}_f : \mathcal{C}_f)_{\text{reg}}$ if and only if $A$ is almost regular and the following two conditions hold

\[
\lim_{q,q' \to \infty} \sum_{i,j} |\Delta_{10} a(i,j,q,q',m,n)| = 0 \text{ uniformly in } m,n \in \mathbb{N}, \tag{28}
\]

\[
\lim_{q,q' \to \infty} \sum_{i} \sum_{j} |\Delta_{01} a(i,j,q,q',m,n)| = 0 \text{ uniformly in } m,n \in \mathbb{N}, \tag{29}
\]

where

\[
\Delta_{10} a(i,j,q,q',m,n) = a(i,j,q,q',m,n) - a(i+1,j,q,q',m,n),
\]

\[
\Delta_{01} a(i,j,q,q',m,n) = a(i,j,q,q',m,n) - a(i,j+1,q,q',m,n).
\]

**Lemma 0.15.** (Yeşilkayagil & Başar, 2016, Theorem 3.5., p.43) The four-dimensional matrix $A = (a_{mnkl}) \in (\mathcal{M}_u : \mathcal{C}_f)$ if and only if the condition (10) and the following conditions hold

\[
\exists \beta_{kl} \in \mathbb{C} \ni f_2 - \lim_{m,n \to \infty} a_{mnkl} = \beta_{kl} \text{ for all } k,l \in \mathbb{N}, \tag{30}
\]

For every $m,n,j \in \mathbb{N}, \exists K \in \mathbb{N} \ni \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} a_{kl} = 0, \tag{31}
\]

for all $q,q', i > K$,

For every $m,n,i \in \mathbb{N}, \exists L \in \mathbb{N} \ni \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} a_{kl} = 0, \tag{32}
\]

for all $q,q', j > L$.

**Lemma 0.16.** (Tug, 2018, Theorem 3., p.4) Let $A = (a_{mnkl})$ be a four-dimensional infinite matrix. Then the following statements hold.

(a) Let $0 < s' \leq 1$. Then, $A = (a_{mnkl}) \in (\mathcal{L}_s' : \mathcal{C}_f)$ if and only if

\[
\sup_{m,n,k,l \in \mathbb{N}} |a_{mnkl}| < \infty, \tag{33}
\]

\[
\exists (\alpha_{kl}) \in \mathbb{C} \text{ such that } f_2 - \lim_{m,n \to \infty} a_{mnkl} = \alpha_{kl} \text{ for all } k,l \in \mathbb{N}. \tag{34}
\]
(b) Let $1 < s' < \infty$. Then, $A = (a_{mnkl}) \in \mathcal{L}_{s'}^{\mathcal{C}_f}$ if and only if the condition (34) holds and
\[
\sup_{m,n \in \mathbb{N}} \sum_{k,l} |a_{mnkl}|^{s'} < \infty. \tag{35}
\]

Lemma 0.17. (Tuğ, 2021, Theorem 4.2., p.13) Four-dimensional matrix $A = (a_{mnkl}) \in (\mathcal{L}_{s'}^{\mathcal{C}_f})$ with $f_2 - \lim Ax = [f_2] - \lim_{kl} x_{kl}$ if and only if $A$ is almost $\mathcal{C}_{bp}$-regular, i.e., $A = (a_{mnkl}) \in (\mathcal{C}_{bp}^{\mathcal{C}_f})$ with $f_2 - \lim Ax = bp - \lim_{kl} x_{kl}$ and
\[
\sum_{k,l \in E} |\Delta_{11} a_{mnkl}| \to 0, \text{ as } m,n \to \infty \tag{36}
\]
for each set $E$ which is uniformly zero density where
\[
\Delta_{11} a_{mnkl} = a_{mnkl} - a_{mn,k+1,l} - a_{mn,k,l+1} + a_{mn,k+1,l+1} \tag{37}
\]

Now we come up with the following corollaries without their proofs.

Corollary 0.18. The four-dimensional matrix $A = (a_{mnkl}) \in (\mathcal{M}_{u}^{\mathcal{C}_f})$ if and only if the condition (10) and the following conditions hold
\[
\exists (\beta_{kl}) \in \mathbb{C} \ni [f_2] - \lim_{m,n \to \infty} a_{mnkl} = \beta_{kl} \text{ for all } k,l \in \mathbb{N}, \tag{38}
\]
For every $m,n,j \in \mathbb{N}$, $\exists K \in \mathbb{N} \ni \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} |a_{klij}| = 0, \tag{39}
\]
for all $q,q',i > K$,
For every $m,n,i \in \mathbb{N}$, $\exists L \in \mathbb{N} \ni \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} |a_{klij}| = 0, \tag{40}
\]
for all $q,q',j > L$.

Corollary 0.19. Let $A = (a_{mnkl})$ be a four-dimensional infinite matrix. Then the following statements hold.

(a) Let $0 < s' \leq 1$. Then, $A = (a_{mnkl}) \in (\mathcal{L}_s^{\mathcal{C}_f})$ if and only if (33) holds and
\[
\exists (a_{kl}) \in \mathbb{C} \text{ such that } [f_2] - \lim_{m,n \to \infty} a_{mnkl} = a_{kl} \text{ for all } k,l \in \mathbb{N} \tag{41}.
\]

(b) Let $1 < s' < \infty$. Then, $A = (a_{mnkl}) \in (\mathcal{L}_{s'}^{\mathcal{C}_f})$ if and only if the condition (41) holds and
\[
\sup_{m,n \in \mathbb{N}} \sum_{k,l} |a_{mnkl}|^{s'} < \infty. \tag{42}
\]

Corollary 0.20. The following statements hold:

(a) A four-dimensional matrix $A = (a_{mnkl})$ is strongly almost $\mathcal{C}_{bp}$-conservative, i.e., $A \in (\mathcal{C}_{bp}^{\mathcal{C}_f})$ if
and only if the condition in (10), and the following conditions hold

\[ \exists a_{ij} \in \mathbb{C} : \exists b_p - \lim_{q,q' \to \infty} \Delta_0 \tilde{a}(i, j, q, q', m, n) = a_{ij} \]

uniformly in \( m, n \in \mathbb{N} \) for each \( i, j \in \mathbb{N} \)

\[ \exists u \in \mathbb{C} : \exists b_p - \lim_{q,q' \to \infty} \sum_{i,j} \tilde{a}(i, j, q, q', m, n) = u \]

uniformly in \( m, n \in \mathbb{N} \)

\[ \exists a_{ij} \in \mathbb{C} : \exists b_p - \lim_{q,q' \to \infty} \sum_{i} |\tilde{a}(i, j, q, q', m, n) - a_{ij}| = 0 \]

uniformly in \( m, n \in \mathbb{N} \) for each \( j \in \mathbb{N} \)

\[ \exists a_{ij} \in \mathbb{C} : \exists b_p - \lim_{q,q' \to \infty} \sum_{j} |\tilde{a}(i, j, q, q', m, n) - a_{ij}| = 0 \]

uniformly in \( m, n \in \mathbb{N} \) for each \( i \in \mathbb{N} \)

where \( \tilde{a}(i, j, q, q', m, n) = \frac{1}{(q+1)(q'+1)} \sum_{k=n}^{m+q} \sum_{l=n}^{q'+q} |a_{klij}| \). In this case, \( a = (a_{ij}) \in \mathcal{L}_a \) and

\[ [f_2] - \lim_{i,j} Ax = \sum_{i,j} a_{ij} x_{ij} + \left( u - \sum_{i,j} a_{ij} \right) b_p - \lim_{i,j \to \infty} x_{ij} \]

that is,

\[ b_p - \lim_{q,q' \to \infty} \sum_{i,j} \tilde{a}(i, j, q, q', m, n) x_{ij} = \sum_{i,j} a_{ij} x_{ij} + \left( u - \sum_{i,j} a_{ij} \right) b_p - \lim_{i,j \to \infty} x_{ij} \]

uniformly in \( m, n \in \mathbb{N} \).

(b) A four-dimensional matrix \( A = (a_{mnkl}) \) is strongly almost \( \mathcal{C}_{bp} \)-regular, i.e., \( A \in (\mathcal{C}_{bp} : [\mathcal{C}_f])_{\text{reg}} \) if and only if the conditions (10), (43)-(46) hold with \( a_{ij} = 0 \) for all \( i, j \in \mathbb{N} \) and \( u = 1 \).

**Corollary 0.21.** Four-dimensional matrix \( A = (a_{mnkl}) \in ([\mathcal{C}_f] : [\mathcal{C}_f]) \) with \( [f_2] - \lim_{i,j} Ax = [f_2] - \lim_{q,q' \to \infty} x_{kl} \) if and only if \( A \) is strongly almost \( \mathcal{C}_{bp} \)-regular, i.e., \( A = (a_{mnkl}) \in (\mathcal{C}_{bp} : [\mathcal{C}_f]) \) with \( [f_2] - \lim_{i,j} Ax = b_p - \lim_{q,q' \to \infty} x_{kl} \) and the condition in (36) holds.

**Corollary 0.22.** A four-dimensional matrix \( A = (a_{mnkl}) \) is strongly almost strongly regular, i.e., \( A \in ([\mathcal{C}_f] : [\mathcal{C}_f])_{\text{reg}} \) if and only if \( A \) is strongly almost regular and the following two conditions hold

\[ \lim_{q,q' \to \infty} \sum_{i,j} \Delta_0 \tilde{a}(i, j, q, q', m, n) = 0 \text{ uniformly in } m, n \in \mathbb{N}, \]  

\[ \lim_{q,q' \to \infty} \sum_{i,j} \Delta_0 \tilde{a}(i, j, q, q', m, n) = 0 \text{ uniformly in } m, n \in \mathbb{N}, \]

where \( \tilde{a}(i, j, q, q', m, n) = \frac{1}{(q+1)(q'+1)} \sum_{k=n}^{m+q} \sum_{l=n}^{q'+q} |a_{klij}| \) and

\[ \Delta_0 \tilde{a}(i, j, q, q', m, n) = \tilde{a}(i, j, q, q', m, n) - \tilde{a}(i + 1, j, q, q', m, n), \]

\[ \Delta_0 \tilde{a}(i, j, q, q', m, n) = \tilde{a}(i, j, q, q', m, n) - \tilde{a}(i, j + 1, q, q', m, n). \]

Now, the following Lemma 0.23 is given by considering four-dimensional dual summability method for double sequences.
Lemma 0.23. (Tuğ et al., 2020, Theorem 13., p.14) Suppose that the four-dimensional matrices $A = (a_{mnkl})$ and $\tilde{C} = (\tilde{c}_{mnkl})$ are connected with the following relation

$$\tilde{c}_{mnkl} = \sum_{i,j,k,l} \frac{1}{\prod_{\pi=k}^{i-1} \prod_{\rho=l}^{j-1} \left( -\frac{s_{\pi}}{r_{\pi}} \right) \left( -\frac{u_{\pi}}{t_{\rho}} \right)} a_{mnij}$$  \hspace{1cm} (49)

for any $m,n,k,l \in \mathbb{N}$ and $\lambda$ is any given double sequence space. Then, $A \in (\tilde{B}(\mu) : \lambda)$ if and only if

$$A_{mn} \in \{\tilde{B}(\mu)\}^{\beta(\delta)}$$ for any $m,n \in \mathbb{N}$, \hspace{1cm} (50)

$$\tilde{C} \in (\mu : \lambda)$$ \hspace{1cm} (51)

Corollary 0.24. Suppose that the four-dimensional matrices $A = (a_{mnkl})$ and $\tilde{C} = (\tilde{c}_{mnkl})$ are connected with the following relation \hspace{1cm} (49). The the followings hold:

(a) $A \in (\tilde{B}(\mathcal{C}_{bp}) : [\mathcal{C}_{f}])$ if and only if the conditions in (50) holds, and (10), (20)-(23) hold with $\tilde{c}_{mnkl}$ instead of $a_{mnkl}$.

(b) $A \in (\tilde{B}(\mathcal{C}_{s}) : [\mathcal{C}_{f}])$ if and only if the conditions in (50) holds, and (10), (20), (21), (24), (25) hold with $\tilde{c}_{mnkl}$ instead of $a_{mnkl}$.

(c) $A \in (\tilde{B}(\mathcal{C}_{p}) : [\mathcal{C}_{f}])$ if and only if the conditions in (50) holds, and (10), (20), (21), (26), (27) hold with $\tilde{c}_{mnkl}$ instead of $a_{mnkl}$.

(d) $A \in (\tilde{B}(\mathcal{L}_{s}) : [\mathcal{C}_{f}])$, $(0 < s' \leq 1)$ if and only if the conditions in (50) holds, and (33) and (34) hold with $\tilde{c}_{mnkl}$ instead of $a_{mnkl}$.

(e) $A \in (\tilde{B}(\mathcal{L}_{s}) : [\mathcal{C}_{f}])$, $(1 < s' < \infty)$ if and only if the conditions in (50) holds, and (34) and (35) hold with $\tilde{c}_{mnkl}$ instead of $a_{mnkl}$.

(f) $A \in (\tilde{B}(\mathcal{C}_{f}) : [\mathcal{C}_{f}])$ if and only if the conditions in (50) holds, and (10), (60)-23 hold with $a_{ij} = 0$ for all $i,j \in \mathbb{N}$ and $u = 1$, and (28) and (29) hold with $\tilde{c}_{mnkl}$ instead of $a_{mnkl}$.

Corollary 0.25. Suppose that the four-dimensional matrices $A = (a_{mnkl})$ and $\tilde{C} = (\tilde{c}_{mnkl})$ are connected with the following relation \hspace{1cm} (49). The the followings hold:

(a) $A \in (\tilde{B}(\mathcal{C}_{bp}) : [\mathcal{C}_{f}])$ if and only if the conditions in (50) holds, and (10), (33)-(46) hold with $\tilde{c}_{mnkl}$ instead of $a_{mnkl}$.

(b) $A \in (\tilde{B}(\mathcal{L}_{s}) : [\mathcal{C}_{f}])$, $(0 < s' \leq 1)$ if and only if the conditions in (50) holds, and (33) and (41) hold with $\tilde{c}_{mnkl}$ instead of $a_{mnkl}$.

(c) $A \in (\tilde{B}(\mathcal{L}_{s}) : [\mathcal{C}_{f}])$, $(1 < s' < \infty)$ if and only if the conditions in (50) holds, and (41) and (42) hold with $\tilde{c}_{mnkl}$ instead of $a_{mnkl}$.

(d) $A \in (\tilde{B}(\mathcal{C}_{f}) : [\mathcal{C}_{f}])$ if and only if the conditions in (50) holds, and (10), (33)-(46) hold with $a_{ij} = 0$ for all $i,j \in \mathbb{N}$ and $u = 1$, and (30)-(32) hold with $\tilde{c}_{mnkl}$ instead of $a_{mnkl}$.

(e) $A \in (\tilde{B}(\mathcal{M}_{u}) : [\mathcal{C}_{f}])$ if and only if the conditions in (50) holds, and (10), (38)-(40) hold with $\tilde{c}_{mnkl}$ instead of $a_{mnkl}$.

(f) $A = (a_{mnkl}) \in (\tilde{B}(\mathcal{C}_{f}) : [\mathcal{C}_{f}])$ with $[f_2] - \lim Ax = [f_2] - \lim_{kl} x_{kl}$ if and only if $A = (a_{mnkl}) \in (\tilde{B}(\mathcal{C}_{bp}) : [\mathcal{C}_{f}])$ with $[f_2] - \lim Ax = bp - \lim_{kl} x_{kl}$ and the condition in (36) holds with $\tilde{c}_{mnkl}$ instead of $a_{mnkl}$.
Corollary 0.26. Suppose that the four-dimensional matrices $A = (a_{mnkl})$ and $\tilde{C} = (\tilde{c}_{mnkl})$ are connected with the following relation (49). The the followings hold:

(a) $A \in (\mathcal{B}(c_f) : \mathcal{C}_g)$ if and only if the conditions in (50) holds, and (10)-(16) hold with $\tilde{c}_{mnkl}$ instead of $a_{mnkl}$.

(b) $A \in (\mathcal{B}(c_f) : \mathcal{M}_u)$ if and only if the conditions in (50) holds, and (10) holds with $\tilde{c}_{mnkl}$ instead of $a_{mnkl}$.

(c) $A \in (\mathcal{B}(\tilde{c}_f) : \mathcal{C}_f)$ if and only if the conditions in (50) holds, and (20)-(23) and (36) hold with $\tilde{c}_{mnkl}$ instead of $a_{mnkl}$.

(d) $A \in (\mathcal{B}(\tilde{c}_f) : \mathcal{C}_b)$ if and only if the conditions in (50) holds, and (11)-(14) hold with $a_{ij} = 0$ for all $i, j \in \mathbb{N}$ and $u = 1$, and (18)-(19) hold with $\tilde{c}_{mnkl}$ instead of $a_{mnkl}$.

(e) $A \in (\mathcal{B}(\tilde{c}_f) : \mathcal{M}_u)$ if and only if the conditions in (50) holds, and (10) holds with $\tilde{c}_{mnkl}$ instead of $a_{mnkl}$.

Lemma 0.27. (Tug et al., 2020, Theorem 14., p.15) Suppose that the four-dimensional infinite matrices $A = (a_{mnkl})$ and $G = (\tilde{g}_{mnkl})$ are connected with the following relation

$$\tilde{g}_{mnkl} = \sum_{i,j=0}^{m,n} b_{mnj} a_{ijkl}$$

(52)

for all $m, n, k, l \in \mathbb{N}$. Then, $A \in (\lambda : \tilde{B}(\mu))$ if and only if

$$\tilde{G} \in (\lambda : \mu)$$

(53)

Corollary 0.28. Suppose that the four-dimensional matrices $A = (a_{mnkl})$ and $\tilde{G} = (\tilde{g}_{mnkl})$ are connected with the following relation (49). The the followings hold:

(a) $A \in (\mathcal{C}_b : \tilde{B}(c_f))$ if and only if the conditions in (10), (20)-(23) hold with $\tilde{g}_{mnkl}$ instead of $a_{mnkl}$.

(b) $A \in (\mathcal{C}_r : \tilde{B}(\tilde{c}_f))$ if and only if the conditions in (10), (20), (21), (24), (25) hold with $\tilde{g}_{mnkl}$ instead of $a_{mnkl}$.

(c) $A \in (\mathcal{C}_p : \tilde{B}(\tilde{c}_f))$ if and only if the conditions in (10), (20), (21), (26), (27) hold with $\tilde{g}_{mnkl}$ instead of $a_{mnkl}$.

(d) $A \in (\mathcal{L}_s : \tilde{B}(\tilde{c}_f))$, $(0 < s' \leq 1)$ if and only if the conditions in (33) and (34) hold with $\tilde{g}_{mnkl}$ instead of $a_{mnkl}$.

(e) $A \in (\mathcal{L}_s : \tilde{B}(c_f))$, $(1 < s' < \infty)$ if and only if the conditions in (34) and (35) hold with $\tilde{g}_{mnkl}$ instead of $a_{mnkl}$.

(f) $A \in (\mathcal{C}_f : \tilde{B}(c_f))$ if and only if the conditions in (10), (20)-(23) hold with $a_{ij} = 0$ for all $i, j \in \mathbb{N}$ and $u = 1$, and (28) and (29) hold with $\tilde{g}_{mnkl}$ instead of $a_{mnkl}$.

Corollary 0.29. Suppose that the four-dimensional matrices $A = (a_{mnkl})$ and $\tilde{G} = (\tilde{g}_{mnkl})$ are connected with the following relation (49). The the followings hold:

(a) $A \in (\mathcal{C}_b : \tilde{B}(c_f))$ if and only if the conditions in (10), (43)-(46) hold with $\tilde{g}_{mnkl}$ instead of $a_{mnkl}$.

(b) $A \in (\mathcal{L}_s : \tilde{B}(\tilde{c}_f))$, $(0 < s' \leq 1)$ if and only if the conditions in (33) and (41) hold with $\tilde{g}_{mnkl}$ instead of $a_{mnkl}$.
(c) $A \in (L_s^2 : \tilde{B}[\mathcal{F}])$, $(1 < s' < \infty)$ if and only if the conditions in (41) and (42) hold with $\tilde{g}_{mnkl}$ instead of $a_{mnkl}$.

(d) $A \in (\mathcal{C}_f : \tilde{B}[\mathcal{F}])$ if and only if the conditions in (10), (43)-(46) hold with $a_{ij} = 0$ for all $i, j \in \mathbb{N}$ and $u = 1$, and (30)-(32) hold with $\tilde{g}_{mnkl}$ instead of $a_{mnkl}$.

(e) $A \in (\mathcal{M}_u : \tilde{B}[\mathcal{F}])$ if and only if the conditions in (10), (38)-(40) hold with $\tilde{g}_{mnkl}$ instead of $a_{mnkl}$.

(f) $A = (a_{mnkl}) \in ([\mathcal{C}_f : \tilde{B}[\mathcal{F}])$ with $[f_2] - \lim Ax = [f_2] - \lim_{kl} x_{kl}$ if and only if $A = (a_{mnkl}) \in ([\mathcal{G}_{bp} : \tilde{B}[\mathcal{F}])$ with $[f_2] - \lim Ax = bp - \lim_{kl} x_{kl}$ and the condition in (36) holds with $\tilde{g}_{mnkl}$ instead of $a_{mnkl}$.

**Corollary 0.30.** Suppose that the four-dimensional matrices $A = (a_{mnkl})$ and $G = (\tilde{g}_{mnkl})$ are connected with the following relation (49). The following hold:

(a) $A \in (\mathcal{C}_f : \tilde{B}(\mathcal{C}_{bp}))$ if and only if (10)-(16) hold with $\tilde{g}_{mnkl}$ instead of $a_{mnkl}$.

(b) $A \in (\mathcal{C}_f : \tilde{B}(\mathcal{M}_u))$ if and only if (10) holds with $\tilde{g}_{mnkl}$ instead of $a_{mnkl}$.

(c) $A \in ([\mathcal{C}_f : \tilde{B}(\mathcal{C}_f)])$ if and only if (20)-(23) and (36) hold with $\tilde{g}_{mnkl}$ instead of $a_{mnkl}$.

(d) $A \in ([\mathcal{C}_f : \tilde{B}(\mathcal{C}_{bp})])$ if and only if (11)-(14) hold with $a_{ij} = 0$ for all $i, j \in \mathbb{N}$ and $u = 1$, and (18)-(19) hold with $\tilde{g}_{mnkl}$ instead of $a_{mnkl}$.

(e) $A \in ([\mathcal{C}_f : \tilde{B}(\mathcal{M}_u)])$ if and only if (10) holds with $\tilde{g}_{mnkl}$ instead of $a_{mnkl}$.

5. Conclusion

Lorentz (1948) introduced almost convergence for single sequence and then M. Mursaleen (2010) investigated the certain properties of the space of almost convergent sequences denoted by $f$. Then many of the mathematician has studied the matrix domain on almost null and almost convergent sequences spaces (see Başar and Kirişçi (2011), Tuğ and Başar (2016), Kayaduman and Şengönül (2012), Şengönül and Kayaduman (2012), RhoadesMoricz and Rhoades (1988) introduced and studied almost convergence for double sequences. Then many significant contributions have been done by several mathematicians (see Tuğ (2018), Tuğ (2018), Tuğ (2021), Mörizc and Rhoades (1990), Mursaleen and Savas (2003), Cunjalo (2007), M. Mursaleen and Mohiuddine (2009), M. Mursaleen and Mohiuddine (2010). In this paper, we defined the spaces $\tilde{B}(\mathcal{C}_f), \tilde{B}(\mathcal{C}_{f_0}), \tilde{B}[\mathcal{C}_f]$ and $\tilde{B}[\mathcal{C}_{f_0}]$, where the matrix $\tilde{B}(\tilde{r}, \tilde{s}, \tilde{i}, \tilde{u})$ was defined by Tuğ et al. (2020) as four-dimensional sequential band matrix. Then we state some topological properties beside some strict inclusion relations. In the third section, we calculated the $\alpha-$, $\beta bp-$ and $\gamma-$ dual spaces of $\tilde{B}(\mathcal{C}_f)$ and $\tilde{B}[\mathcal{F}]$. In the last section, we stated the known matrix classes from or into the spaces $\tilde{B}(\mathcal{C}_f)$ and $\tilde{B}[\mathcal{F}]$, and then we characterized some new matrix classes $(\tilde{B}(\mu : \lambda))$, $(\lambda : \tilde{B}(\mu))$, where $\lambda, \mu$ are any sequence spaces from the set $\{\mathcal{C}_f, \mathcal{C}_f, \mathcal{C}_{bp}, \mathcal{C}_r, \mathcal{C}_p, L_s^2, \mathcal{M}_u\}$.

**References**


