Almost and Strongly Almost $B(\widetilde{r},\widetilde{s},\widetilde{t},\widetilde{u})-$Summable Double Sequences

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Abstract: In this paper, we define some new almost and strongly almost convergent double sequence spaces $\widetilde{B}(\mathscr{C}_f)$, $\widetilde{B}(\mathscr{C}_{f_0})$, $\widetilde{B}[\mathscr{C}_f]$ and $\widetilde{B}[\mathscr{C}_{f_0}]$ derived by the domain of four-dimensional sequential band matrix $B(\widetilde{r},\widetilde{s},\widetilde{t},\widetilde{u})$ in the spaces $\mathscr{C}_f$, $\mathscr{C}_{f_0}$, $[\mathscr{C}_f]$ and $[\mathscr{C}_{f_0}]$, respectively. Then we study some topological properties and prove some strict inclusion relations. Moreover, we calculate the $\alpha-,\beta bp-$ and $\gamma-$duals of the new spaces. Finally, we state some known lemmas concerning the four-dimensional matrix classes of almost convergent double sequences, then we characterize some new four-dimensional matrix transformations from and into the new sequence spaces $\widetilde{B}(\mathscr{C}_f)$ and $\widetilde{B}[\mathscr{C}_f]$. We conclude the paper with several significant results.

Keywords: Four-dimensional band matrix; matrix domain; almost convergence; double sequences; dual spaces; matrix transformations.

1. Introduction and Preliminaries

By the set $\Omega := \{x = (x_{mn}) : x_{mn} \in \mathbb{C}, \ \ \forall m,n \in \mathbb{N}\}$, we denote all complex valued double sequences, where $\mathbb{C}$ is the complex field. Ω is a vector space with coordinatewise addition and scalar multiplication. Any vector subspace of Ω is called a double sequence space. $\mathscr{M}_u$, $\mathscr{C}_p$, $\mathscr{C}_{bp}$, $\mathscr{C}_r$, and $\mathscr{L}_q$ denote the classical spaces of all double sequences that are bounded, convergent in the Pringsheim sense, convergent in the Pringsheim sense and bounded, regular convergent, and q-absolutely summable, respectively, where $0 < q < \infty$. It is well known that the space $\mathscr{L}_q$ becomes the space $\mathscr{L}_u$ in the case $q = 1$. Moreover, by $\mathscr{BS}$, $\mathscr{CS}_\vartheta$, where $\vartheta = \{p, bp, r\}$, we denote all bounded and ϑ-convergent series, respectively.
Let E be any double sequence space. Then,

$$E^{\beta(\vartheta)} := \left\{ a = (a_{kl}) \in \Omega : \{a_{kl}x_{kl}\} \in \mathscr{CS}_\vartheta, \textit{for every } x = (x_{kl}) \in E \right\},$$

$$E^{\alpha} := \left\{ a = (a_{kl}) \in \Omega : \{a_{kl}x_{kl}\} \in \mathscr{L}_u, \textit{for every } x = (x_{kl}) \in E \right\},$$

$$E^{\gamma} := \left\{ a = (a_{kl}) \in \Omega : \{a_{kl}x_{kl}\} \in \mathscr{BS}, \textit{for every } x = (x_{kl}) \in E \right\}.$$

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Therefore, let E_1 and E_2 are arbitrary double sequences with $E_2 \subset E_1$ then the inclusions $E_1^{\alpha} \subset E_2^{\alpha}$, $E_1^{\gamma} \subset E_1^{\alpha}$ and $E_1^{\beta(\vartheta)} \subset E_1^{\alpha}$ hold. But the inclusion $E_1^{\gamma} \subset E_1^{\beta(\vartheta)}$ does not hold, since $\mathscr{C}_p \setminus \mathscr{M}_u$ is not empty.

Let $A = (a_{mnkl})_{m,n,k,l \in \mathbb{N}}$ be an infinite four–dimensional matrix and $E_1, E_2 \in \Omega$. We write

$$y_{mn} = A_{mn}(x) = \vartheta - \sum_{k,l} a_{mnk} x_{kl} \text{ for each } m,n \in \mathbb{N}. \tag{1}$$

We say that A defines a matrix transformation from E_1 to E_2 if

$$A(x) = (A_{mn}(x))_{m,n} \in E_2 \text{ for all } x \in E_1. \tag{2}$$

The $\vartheta-$summability domain $E_A^{(\vartheta)}$ of a four-dimensional infinite matrix A in a double sequence space E is defined by

$$E_A^{(\vartheta)} = \left\{ x = (x_{kl}) \in \Omega : Ax = \left(\vartheta - \sum_{k,l} a_{mnkl} x_{kl} \right)_{m,n \in \mathbb{N}} \textit{exists and is in } E \right\},$$

which is a sequence space. The above notation (2) says that $A = (a_{mnkl})_{m,n,k,l \in \mathbb{N}}$ maps the space E_1 into the space E_2 if $E_1 \subset (E_2)_A^{(\vartheta)}$ and we denote the set of all four-dimensional matrices that map the space E_1 into the space E_2 by $(E_1 : E_2)$. Thus, $A \in (E_1 : E_2)$ if and only if the double series on the right side of (2) $\vartheta-$converges for each $m,n \in \mathbb{N}$, i.e, $A_{mn} \in (E_1)^{\beta(\vartheta)}$ for all $m,n \in \mathbb{N}$ and we have $Ax \in E_2$ for all $x \in E_1$.

Adams (1933) defined that the four-dimensional infinite matrix $A = (a_{mnkl})$ is a triangular matrix if $a_{mnkl} = 0$ for $k > m$ or $l > n$ or both. We also say by Adams (1933) that a triangular matrix $A = (a_{mnkl})$ is called a triangle if $a_{mnmn} \neq 0$ for all $m,n \in \mathbb{N}$. One can be observed easily that if A is triangle, then $E_A^{(\vartheta)}$ and E are linearly isomorphic.

The concept of almost convergence for single sequence introduced by Lorentz (1948) and then Moricz and Rhoades (1988) extended the idea of almost convergence for double sequence. He stated that a double sequence $x = (x_{kl})$ of complex numbers is called almost convergent to a generalized limit L if

$$p - \lim_{q,q' \to \infty} \sup_{m,n>0} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} x_{kl} - L \right| = 0.$$

In this case, L is called the f_2-limit of the double sequence x. Then Başarir (1995) defined the concept of strongly almost convergence of double sequences. A double sequence $x = (x_{kl})$ of real numbers is said to be strongly almost convergent to a limit L_1 if

$$p - \lim_{q,q' \to \infty} \sup_{m,n>0} \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} |x_{kl} - L_1| = 0.$$

and it is uniform in $m,n \in \mathbb{N}$. Now we may define the set of all almost convergent, almost null, strongly almost convergent and strongly almost null double sequences, respectively, as follow;

$$\mathscr{C}_f := \left\{ \begin{array}{c} x = (x_{kl}) \in \Omega : \exists L \in \mathbb{C} \ni p - \lim_{q,q' \to \infty} \sup_{m,n>0} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} x_{kl} - L \right| = 0, \\ \textit{uniformly in } m,n \in \mathbb{N} \textit{ for some } L \end{array} \right\},$$

$$\mathscr{C}_{f_0} := \left\{ \begin{array}{c} x=(x_{kl})\in\Omega : \exists L\in\mathbb{C}\ni p-\lim_{q,q'\to\infty}\sup_{m,n>0}\left|\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}x_{kl}\right|=0, \\ \textit{uniformly in } m,n\in\mathbb{N} \end{array} \right\},$$

$$[\mathscr{C}_f] := \left\{ \begin{array}{c} x=(x_{kl})\in\Omega : \exists L\in\mathbb{C}\ni p-\lim_{q,q'\to\infty}\sup_{m,n>0}\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}|x_{kl}-L_1|=0, \\ \textit{uniformly in } m,n\in\mathbb{N} \textit{ for some } L_1 \end{array} \right\},$$

$$[\mathscr{C}_{f_0}] := \left\{ \begin{array}{c} x=(x_{kl})\in\Omega : \exists L\in\mathbb{C}\ni p-\lim_{q,q'\to\infty}\sup_{m,n>0}\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}|x_{kl}|=0, \\ \textit{uniformly in } m,n\in\mathbb{N} \end{array} \right\},$$

Here we can say for this case that L_1 is called $[f_2]-$limit of a double sequence $x=(x_{kl})$ and written shortly as $[f_2]-\lim x = L_1$.

Here we state some geometrical and topological properties of these sets. Unlike single sequence convergent double sequence need not be almost convergent. But it is well known that every bounded convergent double sequence is also almost convergent and every almost convergent double sequence is bounded. That is, the inclusions $\mathscr{C}_{bp}\subset\mathscr{C}_f\subset\mathscr{M}_u$ strictly hold. Since the following inequality

$$\sup_{m,n>0}\left|\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}x_{kl}-L\right| \le \sup_{m,n>0}\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}|x_{kl}-L|.$$

holds, we can easily say that if a double sequence is strongly almost convergent, that is, the right hand side of the above inequality approaches to zero if we pass to limit as $q,q'\to\infty$, then the left hand side of the inequality also tends to zero. It says that the inclusion $[\mathscr{C}_f]\subset\mathscr{C}_f$ holds and it easily can be seen that the double sequence $x_{kl}=(-1)^l$, for all $k\in\mathbb{N}$, is in $\mathscr{C}_f\setminus[\mathscr{C}_f]$. So the inclusion is strictly hold. Now, we can mention here that the inclusions $\mathscr{C}_{bp}\subset[\mathscr{C}_{f_0}]\subset[\mathscr{C}_f]\subset\mathscr{C}_{f_0}\subset\mathscr{C}_f\subset\mathscr{M}_u$ are strictly hold and each inclusion is proper.

Furthermore, the sets $\mathscr{C}_f$ and $\mathscr{C}_{f_0}$ are Banach spaces with the norm

$$\|x\|_{\mathscr{C}_f} = \sup_{q,q',m,n\in\mathbb{N}}\left|\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}x_{kl}\right|.$$

and the sets $[\mathscr{C}_f]$ and $[\mathscr{C}_{f_0}]$ are Banach spaces with the norm

$$\|x\|_{[\mathscr{C}_f]} = \sup_{q,q',m,n\in\mathbb{N}}\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}|x_{kl}|.$$

The four-dimensional sequential band matrix $B(\widetilde{r},\widetilde{s},\widetilde{t},\widetilde{u})=\{b_{mnkl}(\widetilde{r},\widetilde{s},\widetilde{t},\widetilde{u})\}$ was defined and studied by Tuğ, Rakočević, and Malkowsky (2020) as follows; let $\widetilde{r}=(r_m)_{m=0}^\infty$, $\widetilde{s}=(s_m)_{m=0}^\infty$, $\widetilde{t}=(t_n)_{n=0}^\infty$, and $\widetilde{u}=(u_n)_{n=0}^\infty$ be given sequences of real numbers in the set $c\setminus c_0$. Then,

$$b_{mnkl}(\widetilde{r},\widetilde{s},\widetilde{t},\widetilde{u}) := \left\{ \begin{array}{ccl} r_mt_n & , & (k,l)=(m,n), \\ r_mu_n & , & (k,l)=(m,n-1), \\ s_mt_n & , & (k,l)=(m-1,n), \\ s_mu_n & , & (k,l)=(m-1,n-1), \\ 0 & , & \text{elsewhere}, \end{array} \right.$$

for all $m,n,k,l\in\mathbb{N}$. Therefore, the four-dimensional sequential band matrix $B(\widetilde{r},\widetilde{s},\widetilde{t},\widetilde{u})$-transforms a double sequence $x=(x_{mn})$ into the double sequence $y=(y_{mn})$ as follow;

$$\begin{aligned} y_{mn} := \{B(\widetilde{r},\widetilde{s},\widetilde{t},\widetilde{u})x\}_{mn} &= \sum_{k,l} b_{mnkl}(\widetilde{r},\widetilde{s},\widetilde{t},\widetilde{u})x_{kl} \quad (3) \\ &= s_{m-1}u_{n-1}x_{m-1,n-1}+s_{m-1}t_nx_{m-1,n}+r_mu_{n-1}x_{m,n-1}+r_mt_nx_{mn} \end{aligned}$$

for all $m,n\in\mathbb{N}$. With respect to the above equation in (3), Tuğ et al. (2020) calculated the inverse $B^{-1}(\widetilde{r},\widetilde{s},\widetilde{t},\widetilde{u}) = F(\widetilde{r},\widetilde{s},\widetilde{t},\widetilde{u}) = \{f_{mnkl}(\widetilde{r},\widetilde{s},\widetilde{t},\widetilde{u})\}$ of $B(\widetilde{r},\widetilde{s},\widetilde{t},\widetilde{u})$ which is defined by

$$f_{mnkl}(\widetilde{r},\widetilde{s},\widetilde{t},\widetilde{u}) := \begin{cases} \frac{1}{r_m t_n}\prod_{i=k}^{m-1}\prod_{j=l}^{n-1}\left(\frac{-s_i}{r_i}\right)\left(\frac{-u_j}{t_j}\right) & , \quad 0\le k\le m,\ 0\le l\le n, \\ 0 & , \quad \text{elsewhere}, \end{cases} \tag{4}$$

for all $m,n,k,l\in\mathbb{N}$. Thus, by considering the matrix $B(\widetilde{r},\widetilde{s},\widetilde{t},\widetilde{u})$ and its inverse matrix $F(\widetilde{r},\widetilde{s},\widetilde{t},\widetilde{u})$, and keeping the relation between $x=(x_{mn})$ and $y=(y_{mn})$ in mind, we can observe the following;

$$x_{mn} = \frac{1}{r_m t_n}\sum_{k,l=0}^{m,n}\prod_{i=m-k}^{m-1}\prod_{j=n-l}^{n-1}\left(\frac{-s_i}{r_i}\right)\left(\frac{-u_j}{t_j}\right)y_{m-k,n-l}, \quad \text{for all } m,n\in\mathbb{N}. \tag{5}$$

The double sequence spaces $\widetilde{B}(E)$, where $E=\{\mathscr{C}_p,\mathscr{M}_u,\mathscr{C}_{bp},\mathscr{C}_r,\mathscr{L}_q\}$ and $1\le q<\infty$ were introduced and studied by Tuğ et al. (2020).

Remark 0.1. *Note that in the case $s_m=s,\ r_m=r$ for all $m\in\mathbb{N}$ and $t_n=t,\ u_n=u$ for all $n\in\mathbb{N}$, the four-dimensional sequential band matrix $B(\widetilde{r},\widetilde{s},\widetilde{t},\widetilde{u})$ is reduced to the four-dimensional generalized difference matrix $B(r,s,t,u)$ which was defined by Tuğ and Başar (2016), and studied in Tuğ (2017b),Tuğ (2018),Tug (2018),Tuğ (2021),Tuğ (2017a),Tuğ (2017). Moreover, in the case $s_m=-r_m=1$ for all $m\in\mathbb{N}$ and $t_n=-u_n=-1$ for all $n\in\mathbb{N}$, the four-dimensional sequential band matrix $B(\widetilde{r},\widetilde{s},\widetilde{t},\widetilde{u})$ is reduced to the four-dimensional difference matrix $\Delta(1,-1,1,-1)$ (see Çapan and Başar (2019)). Therefore, the results produced by domain of the matrix $B(\widetilde{r},\widetilde{s},\widetilde{t},\widetilde{u})$ are more comprehensive than the corresponding consequences of the matrix domain of the matrices $B(r,s,t,u)$ and $\Delta(1,-1,1,-1)$.*

2. The new spaces of almost and strongly almost convergent double sequences

In this section, we define spaces $\widetilde{B}(\mathscr{C}_f)$, $\widetilde{B}(\mathscr{C}_{f0})$, $\widetilde{B}[\mathscr{C}_f]$ and $\widetilde{B}[\mathscr{C}_{f0}]$ whose four-dimensional sequential band matrix $B(\widetilde{r},\widetilde{s},\widetilde{t},\widetilde{u})$ domains are in the double sequence spaces $\mathscr{C}_f$, $\mathscr{C}_{f0}$, $[\mathscr{C}_f]$ and $[\mathscr{C}_{f0}]$, respectively. Then we give some topological properties and prove some strict inclusion relations.
Now, we define the spaces $\widetilde{B}(\mathscr{C}_f)$, $\widetilde{B}(\mathscr{C}_{f0})$, $\widetilde{B}[\mathscr{C}_f]$ and $\widetilde{B}[\mathscr{C}_{f0}]$ as follows;

$$\widetilde{B}(\mathscr{C}_f) := \left\{ \begin{array}{c} x=(x_{kl})\in\Omega : \exists L\in\mathbb{C}\ni p-\lim_{q,q'\to\infty}\sup_{m,n>0}\left|\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}(\widetilde{B}x)_{kl}-L\right|=0, \\ \textit{uniformly in } m,n\in\mathbb{N} \textit{ for some } L \end{array} \right\},$$

$$\widetilde{B}(\mathscr{C}_{f0}) := \left\{ \begin{array}{c} x=(x_{kl})\in\Omega : \exists L\in\mathbb{C}\ni p-\lim_{q,q'\to\infty}\sup_{m,n>0}\left|\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}(\widetilde{B}x)_{kl}\right|=0, \\ \textit{uniformly in } m,n\in\mathbb{N} \end{array} \right\},$$

$$\widetilde{B}[\mathscr{C}_f] := \left\{ \begin{array}{c} x=(x_{kl})\in\Omega : \exists L\in\mathbb{C}\ni p-\lim_{q,q'\to\infty}\sup_{m,n>0}\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}\left|(\widetilde{B}x)_{kl}-L\right|=0, \\ \textit{uniformly in } m,n\in\mathbb{N} \textit{ for some } L_1 \end{array} \right\},$$

$$\widetilde{B}[\mathscr{C}_{f0}] := \left\{ \begin{array}{c} x=(x_{kl})\in\Omega : \exists L\in\mathbb{C}\ni p-\lim_{q,q'\to\infty}\sup_{m,n>0}\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}\left|(\widetilde{B}x)_{kl}\right|=0, \\ \textit{uniformly in } m,n\in\mathbb{N}, \end{array} \right\}$$

where $\{\widetilde{B}x\}_{kl} = s_{k-1}u_{l-1}x_{k-1,l-1}+s_{k-1}t_lx_{k-1,l}+r_ku_{l-1}x_{k,l-1}+r_kt_lx_{kl}$.

Theorem 0.1. *The double sequence spaces $\widetilde{B}(\mathscr{C}_f)$, $\widetilde{B}(\mathscr{C}_{f0})$, $\widetilde{B}[\mathscr{C}_f]$ and $\widetilde{B}[\mathscr{C}_{f0}]$ are linearly isomorphic to the spaces $\mathscr{C}_f$, $\mathscr{C}_{f0}$, $[\mathscr{C}_f]$ and $[\mathscr{C}_{f0}]$,respectively; that is, $\widetilde{B}(\mathscr{C}_f)\cong\mathscr{C}_f$, $\widetilde{B}(\mathscr{C}_{f0})\cong\mathscr{C}_{f0}$, $\widetilde{B}[\mathscr{C}_f]\cong[\mathscr{C}_f]$ and $\widetilde{B}[\mathscr{C}_{f0}]\cong[\mathscr{C}_{f0}]$.*

Proof. Since the other cases can be proved similarly, we prove only $\widetilde{B}(\mathscr{C}_f) \cong \mathscr{C}_f$. To prove this, we need to show the existence of a linear bijection between the spaces $\widetilde{B}(\mathscr{C}_f)$ and $\mathscr{C}_f$. Let us define the transformation T from $\widetilde{B}(\mathscr{C}_f)$ to $\mathscr{C}_f$ by $x \mapsto Tx = y = B(\widetilde{r},\widetilde{s},\widetilde{t},\widetilde{u})x$. Linearity of T is clear. Moreover, it can be seen that $x = \theta$ whenever $Tx = \theta$, which means that T is injective.
Let us suppose that an arbitrary $y = (y_{kl}) \in \mathscr{C}_f$ and define $x = (x_{mn})$ via the sequence y by the relation (5) for all $m,n \in \mathbb{N}$. Therefore, we obtain by considering the qualities (3) and (5) that

$$\begin{aligned}
\{B(\widetilde{r},\widetilde{s},\widetilde{t},\widetilde{u})x\}_{mn} &= s_{m-1}u_{n-1}x_{m-1,n-1} + s_{m-1}t_n x_{m-1,n} + r_m u_{n-1}x_{m,n-1} + r_m t_n x_{mn} \\
&= s_{m-1}u_{n-1}\left(\frac{1}{r_{m-1}t_{n-1}}\sum_{k=0}^{m-1}\sum_{l=0}^{n-1}\prod_{i=m-k-1}^{m-2}\prod_{j=n-l-1}^{n-2}\left(\frac{-s_i}{r_i}\right)\left(\frac{-u_j}{t_j}\right)y_{m-k-1,n-l-1}\right) \\
&+ s_{m-1}t_n\left(\frac{1}{r_{m-1}t_n}\sum_{k=0}^{m-1}\sum_{l=0}^{n}\prod_{i=m-k-1}^{m-2}\prod_{j=n-l}^{n-1}\left(\frac{-s_i}{r_i}\right)\left(\frac{-u_j}{t_j}\right)y_{m-k-1,n-l}\right) \\
&+ r_m u_{n-1}\left(\frac{1}{r_m t_{n-1}}\sum_{k=0}^{m}\sum_{l=0}^{n-1}\prod_{i=m-k}^{m-1}\prod_{j=n-l-1}^{n-2}\left(\frac{-s_i}{r_i}\right)\left(\frac{-u_j}{t_j}\right)y_{m-k,n-l-1}\right) \\
&+ r_m t_n\left(\frac{1}{r_m t_n}\sum_{k=0}^{m}\sum_{l=0}^{n}\prod_{i=m-k}^{m-1}\prod_{j=n-l}^{n-1}\left(\frac{-s_i}{r_i}\right)\left(\frac{-u_j}{t_j}\right)y_{m-k,n-l}\right) \\
&= \sum_{k=0}^{m-1}\sum_{l=0}^{n-1}\left(\frac{s_{m-1}u_{n-1}}{r_{m-1}t_{n-1}}\prod_{i=m-k-1}^{m-2}\prod_{j=n-l-1}^{n-2}\left(\frac{-s_i}{r_i}\right)\left(\frac{-u_j}{t_j}\right)y_{m-k-1,n-l-1}\right. \\
&+ \frac{s_{m-1}t_n}{r_{m-1}t_n}\prod_{i=m-k-1}^{m-2}\prod_{j=n-l}^{n-1}\left(\frac{-s_i}{r_i}\right)\left(\frac{-u_j}{t_j}\right)y_{m-k-1,n-l} \\
&+ \frac{r_m u_{n-1}}{r_m t_{n-1}}\prod_{i=m-k}^{m-1}\prod_{j=n-l-1}^{n-2}\left(\frac{-s_i}{r_i}\right)\left(\frac{-u_j}{t_j}\right)y_{m-k,n-l-1} \\
&+ \left.\frac{r_m t_n}{r_m t_n}\prod_{i=m-k}^{m-1}\prod_{j=n-l}^{n-1}\left(\frac{-s_i}{r_i}\right)\left(\frac{-u_j}{t_j}\right)y_{m-k,n-l}\right) + \sum_{k=0}^{m-1}\prod_{i=m-k}^{m-2}\prod_{j=0}^{n-1}\left(\frac{-s_i}{r_i}\right)\left(\frac{-u_j}{t_j}\right)y_{m-k-1,0} \\
&+ \sum_{l=0}^{n-1}\prod_{i=0}^{m-1}\prod_{j=n-l}^{n-2}\left(\frac{-s_i}{r_i}\right)\left(\frac{-u_j}{t_j}\right)y_{0,n-l-1} + \sum_{l=0}^{n-1}\prod_{i=0}^{m-1}\prod_{j=n-l}^{n-1}\left(\frac{-s_i}{r_i}\right)\left(\frac{-u_j}{t_j}\right)y_{0,n-l} \\
&+ \sum_{k=0}^{m-1}\prod_{i=0}^{m-1}\prod_{j=0}^{n-1}\left(\frac{-s_i}{r_i}\right)\left(\frac{-u_j}{t_j}\right)y_{m-k,0} + \prod_{i=0}^{m-1}\prod_{j=0}^{n-1}\left(\frac{-s_i}{r_i}\right)\left(\frac{-u_j}{t_j}\right)y_{00} \\
&= y_{mn}
\end{aligned}$$

for all $m,n \in \mathbb{N}$. This equality leads us to the fact that

$$p-\lim_{q,q'\to\infty}\sup_{m,n>0}\left|\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}(\widetilde{B}x)_{kl}\right| = p-\lim_{q,q'\to\infty}\sup_{m,n>0}\left|\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}y_{kl}\right|.$$

This shows us that $x = (x_{mn}) \in \widetilde{B}(\mathscr{C}_f)$ since $y = (y_{mn})$ lies in $\mathscr{C}_f$. Thus, T is surjective. Therefore, T is a linear bijection between the spaces $\widetilde{B}(\mathscr{C}_f)$ and $\mathscr{C}_f$. It completes the proof. $\square$

Theorem 0.2. *The following inclusions strictly hold.*

(i) $\mathscr{C}_f \subset \widetilde{B}(\mathscr{C}_f)$.

(ii) $\mathscr{C}_{f_0} \subset \widetilde{B}(\mathscr{C}_{f_0})$.

(iii) $[\mathscr{C}_f] \subset \widetilde{B}[\mathscr{C}_f]$.

(iv) $[\mathscr{C}_{f_0}] \subset \widetilde{B}[\mathscr{C}_{f_0}]$.

(v) $\widetilde{B}[\mathscr{C}_f] \subset \widetilde{B}(\mathscr{C}_f)$.

(vi) $\widetilde{B}[\mathscr{C}_{f_0}] \subset \widetilde{B}(\mathscr{C}_{f_0})$.

Proof. $(i-ii)$: Suppose that $x=(x_{kl})$ be a double sequence in the set $\mathscr{C}_f$ such that

$$p-\lim_{q,q'\to\infty}\sup_{m,n>0}\left|\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}x_{kl}\right|$$

exists uniformly in $m,n\in\mathbb{N}$. We need to show that the double sequence $x=(x_{kl})$ is also in $\widetilde{B}(\mathscr{C}_f)$, i.e., $\widetilde{B}x\in\mathscr{C}_f$. Thus, we have

$$\begin{aligned}
&\left|\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}(Bx)_{kl}\right| \\
= &\left|\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}(s_{k-1}u_{l-1}x_{k-1,l-1}+s_{k-1}t_lx_{k-1,l}+r_ku_{l-1}x_{k,l-1}+r_kt_lx_{kl})\right| \\
\leq &\left|\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}(s_{k-1}u_{l-1}x_{k-1,l-1})\right|+\left|\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}(s_{k-1}t_lx_{k-1,l})\right| \\
+ &\left|\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}(r_ku_{l-1}x_{k,l-1})\right|+\left|\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}(r_kt_lx_{kl})\right|.
\end{aligned}$$

Hence, if we pass $p-$limit by letting $q,q'\to\infty$, then

$$p-\lim_{q,q'\to\infty}\sup_{m,n>0}\left|\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}(\widetilde{B}x)_{kl}\right| \tag{6}$$

exists uniformly in $m,n\in\mathbb{N}$ which means that the inclusion $\mathscr{C}_f\subset\widetilde{B}(\mathscr{C}_f)$ holds.
Now, we must show that the set $\widetilde{B}(\mathscr{C}_f)\setminus\mathscr{C}_f$ is not empty. Let us define $x=(x_{kl})$ by

$$x_{kl}=\frac{1}{r_kt_l}\prod_{i=0}^{k-1}\prod_{j=0}^{l-1}\left(\frac{-s_i}{r_i}\right)\left(\frac{-u_j}{t_j}\right) \tag{7}$$

for all $k,l\in\mathbb{N}$. Clearly $x=(x_{kl})$ is not in $\mathscr{C}_f$. Furthermore,

$$\begin{aligned}
(\widetilde{B}x)_{kl} &= s_{k-1}u_{l-1}x_{k-1,l-1}+s_{k-1}t_lx_{k-1,l}+r_ku_{l-1}x_{k,l-1}+r_kt_lx_{kl} \\
&= s_{k-1}u_{l-1}\frac{1}{r_{k-1}t_{l-1}}\prod_{i=0}^{k-2}\prod_{j=0}^{l-2}\left(\frac{-s_i}{r_i}\right)\left(\frac{-u_j}{t_j}\right)+s_{k-1}t_l\frac{1}{r_{k-1}t_l}\prod_{i=0}^{k-2}\prod_{j=0}^{l-1}\left(\frac{-s_i}{r_i}\right)\left(\frac{-u_j}{t_j}\right) \\
&+ r_ku_{l-1}\frac{1}{r_kt_{l-1}}\prod_{i=0}^{k-1}\prod_{j=0}^{l-2}\left(\frac{-s_i}{r_i}\right)\left(\frac{-u_j}{t_j}\right)+r_kt_l\frac{1}{r_kt_l}\prod_{i=0}^{k-1}\prod_{j=0}^{l-1}\left(\frac{-s_i}{r_i}\right)\left(\frac{-u_j}{t_j}\right) \\
&= 0.
\end{aligned}$$

that is, $\widetilde{B}x=0$ for every $k,l\in\mathbb{N}$, which is in the space $\mathscr{C}_f$, i.e., $x\in\widetilde{B}(\mathscr{C}_f)\setminus\mathscr{C}_f$. This gives us that the inclusion $\mathscr{C}_f\subset\widetilde{B}(\mathscr{C}_f)$ is strict.
$(iii-iv)$: Similarly, let us assume that $x=(x_{kl})$ be a double sequence in the space $[\mathscr{C}_f]$ such that

$$p-\lim_{q,q'\to\infty}\sup_{m,n>0}\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}|x_{kl}|$$

exists uniformly in $m,n \in \mathbb{N}$. We must show here that the double sequence $x = (x_{kl})$ is also in $\widetilde{B}[\mathscr{C}_f]$, i.e., $\widetilde{B}x \in [\mathscr{C}_f]$. Thus, we have

$$\begin{aligned}
& \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} |(Bx)_{kl}| \\
= & \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} |s_{k-1}u_{l-1}x_{k-1,l-1} + s_{k-1}t_l x_{k-1,l} + r_k u_{l-1} x_{k,l-1} + r_k t_l x_{kl}| \\
\leq & \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} |s_{k-1}u_{l-1}x_{k-1,l-1}| + \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} |s_{k-1}t_l x_{k-1,l}| \\
+ & \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} |r_k u_{l-1} x_{k,l-1}| + \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} |r_k t_l x_{kl}| \\
\leq & \frac{\|s\|_\infty \|u\|_\infty}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} |x_{k-1,l-1}| + \frac{\|s\|_\infty \|t\|_\infty}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} |x_{k-1,l}| \\
+ & \frac{\|r\|_\infty \|u\|_\infty}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} |x_{k,l-1}| + \frac{\|r\|_\infty \|t\|_\infty}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} |x_{kl}|.
\end{aligned}$$

Hence, if we pass $p-$limit by letting $q,q' \to \infty$, then

$$p - \lim_{q,q' \to \infty} \sup_{m,n>0} \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} \left| (\widetilde{B}x)_{kl} \right| \tag{8}$$

exists uniformly in $m,n \in \mathbb{N}$. Thus, the inclusion $[\mathscr{C}_f] \subset \widetilde{B}[\mathscr{C}_f]$ holds.
Now, to show that the set $\widetilde{B}[\mathscr{C}_f] \setminus [\mathscr{C}_f]$ is not empty, we may consider the double sequence $x = (x_{kl})$ defined as (7) such that $\widetilde{B}x = 0$ which is also in the space $[\mathscr{C}_f]$, i.e., $x \in \widetilde{B}[\mathscr{C}_f] \setminus [\mathscr{C}_f]$. This gives us that the inclusion $[\mathscr{C}_f] \subset \widetilde{B}[\mathscr{C}_f]$ is strict.
$(v - vi)$: The proofs can be proved easily since $[\mathscr{C}_f] \subset \mathscr{C}_f$ and $[\mathscr{C}_{f_0}] \subset \mathscr{C}_{f_0}$ and the double sequence $x = (x_{kl})$ defined as in (7) shows that the sets $\mathscr{C}_f \setminus [\mathscr{C}_f]$ and $\mathscr{C}_{f_0} \setminus [\mathscr{C}_{f_0}]$ are not empty. So, we omit the details. □

Theorem 0.3. *The following statements hold.*

(a) If $\frac{\sup_m s_m}{\inf_m r_m} < 1$ for all $m \in \mathbb{N}$ and $\frac{\sup_n u_n}{\inf_n t_n} < 1$ for all $n \in \mathbb{N}$. The spaces $\mathscr{M}_u$ and μ do not contain each other where $\mu = \{\widetilde{B}[C_f], \widetilde{B}[C_{f_0}]\}$.

(b) If $\frac{\sup_m s_m}{\inf_m r_m} < 1$ for all $m \in \mathbb{N}$ and $\frac{\sup_n u_n}{\inf_n t_n} < 1$ for all $n \in \mathbb{N}$. Then, the following inclusions $\widetilde{B}(\mathscr{C}_f) \subset \mathscr{M}_u$ and $\widetilde{B}(\mathscr{C}_{f_0}) \subset \mathscr{M}_u$ strictly hold.

Proof. (a) : To prove this claim, we must prove that the sets $\widetilde{B}[\mathscr{C}_{f_0}] \setminus \mathscr{M}_u$, $\mathscr{M}_u \setminus \widetilde{B}[\mathscr{C}_{f_0}]$ and $\widetilde{B}[\mathscr{C}_{f_0}] \cap \mathscr{M}_u$ are not empty. Let us define a double sequence $x = (x_{kl})$ by

$$x_{kl} = \frac{k(-1)^l}{r_k t_l} \text{ for all } k,l \in \mathbb{N}.$$

Thus, $\widetilde{B}x \in [\mathscr{C}_{f_0}]$ since $\frac{\sup_m s_m}{\inf_m r_m} < 1$ for all $m \in \mathbb{N}$ and $\frac{\sup_n u_n}{\inf_n t_n} < 1$ for all $n \in \mathbb{N}$, but clearly not in the space $\mathscr{M}_u$. The set $\widetilde{B}[\mathscr{C}_{f_0}] \setminus \mathscr{M}_u$ is not empty. Moreover, if we define a double sequence $x^{(1)} = (x^{(1)}_{kl})$ by $x^{(1)} = e$ then, it is clear that $x^{(1)} \in \widetilde{B}[\mathscr{C}_{f_0}] \cap \mathscr{M}_u$
Now, if we define a double sequence $x^{(2)}$ which is in the space $\mathscr{M}_u$ by

$$x^{(2)}_{kl} = \begin{cases} \frac{1}{r_k t_l} & , \text{ if both } k \text{ and } l \text{ are even} \\ 0 & , \text{ otherwise.} \end{cases}$$

Then, we have the $\widetilde{B}-$transform of $x = (x^{(2)}_{kl})$ as

$$(\widetilde{B}x)_{kl} = \begin{cases} \frac{s_{k-1}u_{l-1}}{r_{k-1}t_{l-1}} & , \quad \textit{if } k \textit{ is odd } l \textit{ is odd}, \\ \frac{s_{k-1}t_l}{r_{k-1}t_l} & , \quad \textit{if } k \textit{ is odd } l \textit{ is even}, \\ \frac{r_k u_{l-1}}{r_k t_{l-1}} & , \quad \textit{if } k \textit{ is even } l \textit{ is odd}, \\ \frac{r_k t_l}{r_k t_l} & , \quad \textit{if } k \textit{ is even } l \textit{ is even}. \end{cases}$$

Thus, one can obtain since $\frac{\sup_m s_m}{\inf_m r_m} < 1$ for all $m \in \mathbb{N}$ and $\frac{\sup_n u_n}{\inf_n t_n} < 1$ for all $n \in \mathbb{N}$ that

$$p - \lim_{q,q'\to\infty} \sup_{m,n>0} \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} \left|(\widetilde{B}x)_{kl}\right| = 1$$

uniformly in $m,n \in \mathbb{N}$. We read from the last approach that $X^{(2)} \in \mathscr{M}_u \setminus \widetilde{B}[\mathscr{C}_{f_0}]$. This is what we claimed.

(b) : First, we need to show that the inclusions $\widetilde{B}(\mathscr{C}_f) \subset \mathscr{M}_u$ and $\widetilde{B}(\mathscr{C}_{f_0}) \subset \mathscr{M}_u$ hold. Let us take any double sequence $x = (x_{kl}) \in \widetilde{B}(\mathscr{C}_f)$. Then $y = \widetilde{B}x \in \mathscr{C}_f \subset \mathscr{M}_u$. Since $\widetilde{B}^{-1}(\widetilde{r},\widetilde{s},\widetilde{t},\widetilde{u})$ satisfy the conditions of (Tuğ, 2018, Theorem 4.10), it belongs to class $(\mathscr{C}_f : \mathscr{M}_u)$ and $x = \widetilde{B}^{-1}y \in \mathscr{M}_u$ holds. Therefore, the inclusion $\widetilde{B}(\mathscr{C}_f) \subset \mathscr{M}_u$ holds.

Now, for the converse, let the following double sequence $x = (x_{kl})$ be defined as

$$x_{kl} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{9}$$

i.e., in each row, there is one 1, then two $0s$, then four $1s$, then eight $0s$, then sixteen $1s$, and goes on with respect to this harmony. Thus, clearly the double sequence $x = (x_{kl}) \in \mathscr{M}_u \setminus \widetilde{B}(\mathscr{C}_f)$. □

3. The $\alpha-, \beta(bp)-$ *and* $\gamma-$duals of the spaces $\widetilde{B}(\mathscr{C}_f)$ and $\widetilde{B}[\mathscr{C}_f]$

In this section, first we calculate the $\alpha-$dual of the spaces $\widetilde{B}(\mathscr{C}_f)$ and $\widetilde{B}[\mathscr{C}_f]$. Then we state some known lemmas concerning the matrix classes $(\mathscr{C}_f : \mathscr{C}_{bp})$, $([\mathscr{C}_f] : \mathscr{C}_{bp})$, $(\mathscr{C}_f : \mathscr{M}_u)$ and $([\mathscr{C}_f] : \mathscr{M}_u)$ which we consider them to calculate the $\beta(bp)-$ and $\gamma-$duals of the spaces $\widetilde{B}(\mathscr{C}_f)$ and $\widetilde{B}[\mathscr{C}_{f_0}]$.

Theorem 0.4. *Let* $\left|\frac{\sup_m s_m}{\inf_m r_m}\right| < 1$ *for all* $m \in \mathbb{N}$ *and* $\left|\frac{\sup_n u_n}{\inf_n t_n}\right| < 1$ *for all* $n \in \mathbb{N}$. *Then, the* $\alpha-$*dual of the spaces* $\widetilde{B}(\mathscr{C}_f)$ *and* $\widetilde{B}[\mathscr{C}_f]$ *is the space* $\mathscr{L}_u$.

Proof. Since the proofs are similar to each other, we only prove here $\left\{\widetilde{B}(\mathscr{C}_f)\right\}^{\alpha} = \mathscr{L}_u$ and leave the other assumption to the reader. To prove our claim $\left\{\widetilde{B}(\mathscr{C}_f)\right\}^{\alpha} = \mathscr{L}_u$, we must show the existence of the inclusions $\mathscr{L}_u \subset \left\{\widetilde{B}(\mathscr{C}_f)\right\}^{\alpha}$ and $\left\{\widetilde{B}(\mathscr{C}_f)\right\}^{\alpha} \subset \mathscr{L}_u$.

For the first inclusion, suppose a sequence $a = (a_{mn}) \in \mathscr{L}_u$ and $x = (x_{mn}) \in \widetilde{B}(\mathscr{C}_f)$. Therefore, there exists a double sequence $y = (y_{mn}) \in \mathscr{C}_f$ with the relation (3) and (5) such that

$$p - \lim_{q,q'\to\infty} \sup_{m,n>0} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} y_{kl} \right|$$

exists. Moreover, the inclusion $\mathscr{C}_f \subset \mathscr{M}_u$ holds, says, $\sup_{m,n\in\mathbb{N}} |y_{mn}| \leq K$ where $K \in \mathbb{R}^+$. Since $\left|\frac{\sup_m s_m}{\inf_m r_m}\right| < 1$ for all $m \in \mathbb{N}$ and $\left|\frac{\sup_n u_n}{\inf_n t_n}\right| < 1$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{m,n}|a_{mn}x_{mn}| &= \sum_{m,n}|a_{mn}|\left|\frac{1}{r_mt_n}\sum_{k,l=0}^{m,n}\prod_{i=m-k}^{m-1}\prod_{j=n-l}^{n-1}\left(\frac{-s_i}{r_i}\right)\left(\frac{-u_j}{t_j}\right)y_{m-k,n-l}\right|\\
&\le \frac{1}{\|\widetilde{r}\|_\infty\|\widetilde{t}\|_\infty}\sum_{m,n}|a_{mn}|\sum_{k,l=0}^{m,n}\left|\prod_{i=m-k}^{m-1}\prod_{j=n-l}^{n-1}\left(\frac{-s_i}{r_i}\right)\left(\frac{-u_j}{t_j}\right)\right||y_{m-k,n-l}|\\
&\le \frac{K}{\|\widetilde{r}\|_\infty\|\widetilde{t}\|_\infty}\sum_{m,n}|a_{mn}|\sum_{k,l=0}^{m,n}\left|\frac{-\sup_m s_m}{\inf_m r_m}\right|^{m-k}\left|\frac{-\sup_n u_n}{\inf_n t_n}\right|^{n-l}\\
&= \frac{K}{\|\widetilde{r}\|_\infty\|\widetilde{t}\|_\infty}\left(\frac{1}{1-\left|\frac{\sup_m s_m}{\inf_m r_m}\right|}\right)\left(\frac{1}{1-\left|\frac{-\sup_n u_n}{\inf_n t_n}\right|}\right)\sum_{m,n}|a_{mn}|\\
&< \infty
\end{aligned}$$

Thus, $a=(a_{mn})\in\left\{\widetilde{B}(\mathscr{M}_u)\right\}^\alpha$ and we can say the fact that the inclusion $\mathscr{L}_u\subset\left\{\widetilde{B}(\mathscr{C}_f)\right\}^\alpha$ holds. Now, to prove the second inclusion $\left\{\widetilde{B}(\mathscr{C}_f)\right\}^\alpha\subset\mathscr{L}_u$, we use converse of this assumption. Let suppose that there exists a sequence $(a_{mn})\in\left\{\widetilde{B}(\mathscr{C}_f)\right\}^\alpha\setminus\mathscr{L}_u$ such that $\sum_{m,n}|a_{mn}x_{mn}|<\infty$ for all $x=(x_{mn})\in\widetilde{B}(\mathscr{C}_f)$. Let the double sequence $x=(x_{kl})$ be defined as $x=(x_{mn})=\{(-1)^{m+n}\}$ which is in $\widetilde{B}(\mathscr{C}_f)$ such that

$$\sum_{m,n}|a_{mn}x_{mn}|=\sum_{m,n}|a_{mn}|=\infty.$$

Thus this is a contradiction. Therefore, (a_{mn}) must belong to the space $\mathscr{L}_u$. It completes the proof. □

Lemma 0.5. *(Moricz & Rhoades, 1988, Theorem 1., p.285) The following statements hold:*

(a) A four-dimensional matrix $A=(a_{mnkl})\in(\mathscr{C}_f:\mathscr{C}_{bp})$ if and only if the following conditions hold:

$$\sup_{m,n\in\mathbb{N}}\sum_{k,l}|a_{mnkl}|<\infty \tag{10}$$

$$\exists a_{kl}\in\mathbb{C}\ni,bp-\lim_{m,n\to\infty}a_{mnkl}=a_{kl}\text{ for all }k,l\in\mathbb{N}, \tag{11}$$

$$\exists u\in\mathbb{C}\ni,bp-\lim_{m,n\to\infty}\sum_{k,l}a_{mnkl}=u, \tag{12}$$

$$\exists k_0\in\mathbb{N}\ni,bp-\lim_{m,n\to\infty}\sum_{l}|a_{mn,k_0,l}-a_{k_0,l}|=0\text{ for all }l\in\mathbb{N}, \tag{13}$$

$$\exists l_0\in\mathbb{N}\ni,bp-\lim_{m,n\to\infty}\sum_{k}|a_{mnk,l_0}-a_{k,l_0}|=0\text{ for all }k\in\mathbb{N}, \tag{14}$$

$$bp-\lim_{m,n\to\infty}\sum_k\sum_l|\Delta_{01}a_{mnkl}|=0, \tag{15}$$

$$bp-\lim_{m,n\to\infty}\sum_k\sum_l|\Delta_{10}a_{mnkl}|=0, \tag{16}$$

where

$$\Delta_{10}a_{mnkl}=a_{mnkl}-a_{mn,k+1,l},\ \ \Delta_{01}a_{mnkl}=a_{mnkl}-a_{mnk,l+1}. \tag{17}$$

(b) A four-dimensional matrix $A=(a_{mnkl})$ is strongly regular, i.e., $A\in(\mathscr{C}_f:\mathscr{C}_{bp})_{reg}$ if and only if the conditions (10)-(16)hold with $a_{kl}=0$ for all $k,l\in\mathbb{N}$ and $u=1$.

where $\Delta_{10}a_{mnkl}=a_{mnkl}-a_{m,n,k+1,l}$ and $\Delta_{01}a_{mnkl}=a_{mnkl}-a_{m,n,k,l+1}$, $(m,n,k,l=0,1,2,\ldots)$.

Lemma 0.6. *(Başarir, 1995, Theorem 1., p.179) A four-dimensional matrix $A = (a_{mnkl} \in ([\mathscr{C}_f] : \mathscr{C}_{bp})$ if and only if A is bounded regular, i.e., $A = (a_{mnkl}) \in (\mathscr{C}_{bp} : \mathscr{C}_{bp})$ (see (Robison, 1926, Theorem I., p.53)), that is the conditions in (11)-(14) hold with $a_{kl} = 0$ for all $k,l \in \mathbb{N}$ and $u = 1$, and satisfy the following two conditions;*

$$bp - \lim_{m,n\to\infty} \sum_{k,l\in E} |\Delta_{10} a_{mnkl}| = 0, \tag{18}$$

$$bp - \lim_{m,n\to\infty} \sum_{k,l\in E} |\Delta_{01} a_{mnkl}| = 0. \tag{19}$$

for each set E which is uniformly zero density

Lemma 0.7. *(Tuğ, 2018, Theorem 4.10, p.14) A four-dimensional matrix $A = (a_{mnkl}) \in (\mathscr{C}_f : \mathscr{M}_u)$ if and only if $A_{mn} \in \{\mathscr{C}_f\}^{\beta(\vartheta)}$ and condition (10) hold.*

Lemma 0.8. *(Tuğ, 2021, Corollary 3.4, p.13) A four-dimensional matrix $A = (a_{mnkl}) \in ([\mathscr{C}_f] : \mathscr{M}_u)$ if and only if the $A_{mn} \in \{[\mathscr{C}_f]\}^{\beta(\vartheta)}$ for all $m,n \in \mathbb{N}$ and (10) holds.*

Now let us define the sets $\widetilde{b}_k$ where $k \in \{1,2,\ldots,7\}$, as follows:

$$\widetilde{b}_1 = \left\{ a = (a_{kl}) \in \Omega : \sup_{m,n\in\mathbb{N}} \sum_{k,l} \left| \sum_{i,j=k,l}^{m,n} \frac{1}{r_i t_j} \prod_{\pi=k}^{i-1} \prod_{\rho=l}^{j-1} \left(\frac{-s_\pi}{r_\pi}\right)\left(\frac{-u_\rho}{t_\rho}\right) a_{ij} \right| < \infty \right\},$$

$$\widetilde{b}_2 = \left\{ a = (a_{kl}) \in \Omega : \exists \beta_{kl} \in \mathbb{C} \ni, \vartheta - \lim_{m,n\to\infty} \sum_{i,j=k,l}^{m,n} \frac{1}{r_i t_j} \prod_{\pi=k}^{i-1} \prod_{\rho=l}^{j-1} \left(\frac{-s_\pi}{r_\pi}\right)\left(\frac{-u_\rho}{t_\rho}\right) a_{ij} = \beta_{kl} \right\},$$

$$\widetilde{b}_3 = \left\{ a = (a_{kl}) \in \Omega : \exists u \in \mathbb{C} \ni, \vartheta - \lim_{m,n\to\infty} \sum_{i,j=k,l}^{m,n} \frac{1}{r_i t_j} \prod_{\pi=k}^{i-1} \prod_{\rho=l}^{j-1} \left(\frac{-s_\pi}{r_\pi}\right)\left(\frac{-u_\rho}{t_\rho}\right) a_{ij} = u \right\},$$

$$\widetilde{b}_4 = \left\{ a = (a_{kl}) \in \Omega : \right.$$

$$\left. \exists l_0 \in \mathbb{N} \ni, \vartheta - \lim_{m,n\to\infty} \sum_{k} \left| \sum_{i,j=k,l_0}^{m,n} \frac{1}{r_i t_j} \prod_{\pi=k}^{i-1} \prod_{\rho=l_0}^{j-1} \left(\frac{-s_\pi}{r_\pi}\right)\left(\frac{-u_\rho}{t_\rho}\right) a_{ij} - \beta_{k,l_0} \right| = 0 \text{ for all } k \in \mathbb{N} \right\},$$

$$\widetilde{b}_5 = \left\{ a = (a_{kl}) \in \Omega : \right.$$

$$\left. \exists k_0 \in \mathbb{N} \ni, \vartheta - \lim_{m,n\to\infty} \sum_{l} \left| \sum_{i,j=k_0,l}^{m,n} \frac{1}{r_i t_j} \prod_{\pi=k_0}^{i-1} \prod_{\rho=l}^{j-1} \left(\frac{-s_\pi}{r_\pi}\right)\left(\frac{-u_\rho}{t_\rho}\right) a_{ij} - \beta_{k_0,l} \right| = 0 \text{ for all } l \in \mathbb{N} \right\},$$

$$\widetilde{b}_6 = \left\{ a = (a_{kl}) \in \Omega : \vartheta - \lim_{m,n\to\infty} \sum_k \sum_l \left| \Delta_{01} \left\{ \sum_{i,j=k,l}^{m,n} \frac{1}{r_i t_j} \prod_{\pi=k}^{i-1} \prod_{\rho=l}^{j-1} \left(\frac{-s_\pi}{r_\pi}\right)\left(\frac{-u_\rho}{t_\rho}\right) a_{ij} \right\} \right| = 0 \right\},$$

$$\widetilde{b}_7 = \left\{ a = (a_{kl}) \in \Omega : \vartheta - \lim_{m,n\to\infty} \sum_k \sum_l \left| \Delta_{10} \left\{ \sum_{i,j=k,l}^{m,n} \frac{1}{r_i t_j} \prod_{\pi=k}^{i-1} \prod_{\rho=l}^{j-1} \left(\frac{-s_\pi}{r_\pi}\right)\left(\frac{-u_\rho}{t_\rho}\right) a_{ij} \right\} \right| = 0 \right\}.$$

Theorem 0.9. *The $\beta(bp)-$dual $\{\mu\}^{\beta(bp)}$ of the space μ is $\bigcap_{i=1}^{7} \widetilde{b}_i$, i.e., $\{\mu\}^{\beta(bp)} = \bigcap_{i=1}^{7} \widetilde{b}_i$, where $\mu = \{\widetilde{B}(\mathscr{C}_f), \widetilde{B}[\mathscr{C}_f]\}$.*

Proof. Suppose that $a = (a_{mn}) \in \Omega$ and $x = (x_{mn}) \in \widetilde{B}(\mathscr{C}_f)$. Then, we have $y = \widetilde{B}x \in \mathscr{C}_f$. m,n-th partial sum of the series $\sum_{k,l} a_{kl} x_{kl}$ is given by the equality (26) which was defined by Tuğ at al (Tuğ et al., 2020, Lemma 1., p.11) and the four-dimensional matrix $\widetilde{D} = (\widetilde{d}_{mnkl})$ which was also defined by Tuğ at al (Tuğ et al., 2020, p. 11) as

$$\widetilde{d}_{mnkl} = \begin{cases} \sum_{i,j=k,l}^{m,n} \frac{1}{r_i t_j} \prod_{\pi=k}^{i-1} \prod_{\rho=l}^{j-1} \left(\frac{-s_\pi}{r_\pi}\right)\left(\frac{-u_\rho}{t_\rho}\right) a_{ij} & , \quad 0 \le k \le m, 0 \le l \le n; \\ 0 & , \quad \text{elsewhere} \end{cases}$$

for all $m,n,k,l \in \mathbb{N}$. Then, since it is given in the hypothesis, one can obtain that $ax \in \mathscr{CS}_{bp}$ whenever $x = (x_{mn}) \in \widetilde{B}(\mathscr{C}_f)$ if and only if $\widetilde{D}y \in \mathscr{C}_{bp}$ whenever $y = (y_{mn}) \in \mathscr{C}_f$. This says that $a = (a_{mn}) \in \left\{\widetilde{B}(\mathscr{C}_f)\right\}^{\beta(bp)}$ if and only if $\widetilde{D} \in (\mathscr{C}_f : \mathscr{C}_{bp})$. Thus, we can say that the conditions of Lemma 0.5(a) holds with $\widetilde{d}_{mnkl}$ instead of a_{mnkl}, i.e., this is the set $\bigcap_{i=1}^{7} \widetilde{d}_i$. This completes the proof.
The $\beta bp-$dual $\{\widetilde{B}[\mathscr{C}_f]\}^{\beta(bp)}$ of the space $\widetilde{B}[\mathscr{C}_f]$ can be calculated by considering $k,l \in E$ which is uniformly zero density in Lemma 0.6 and then it is seen that $\{\widetilde{B}[\mathscr{C}_f]\}^{\beta(bp)} = \bigcap_{i=1}^{7} \widetilde{d}_i$ too. So we omit the repetition. □

Theorem 0.10. *The $\gamma-$dual of the spaces $\widetilde{B}(\mathscr{C}_f)$ and $\widetilde{B}[\mathscr{C}_f]$ is the set $\widetilde{d}_1 \cap CS_{\vartheta}$, i.e., $\{\widetilde{B}(\mathscr{C}_f)\}^{\gamma} = \{\widetilde{B}[\mathscr{C}_f]\}^{\gamma} = \widetilde{d}_1 \cap CS_{\vartheta}$*

Proof. Suppose that $a = (a_{mn}) \in \Omega$ and $x = (x_{mn}) \in \widetilde{B}[\mathscr{C}_f]$. We need to show that the $(m,n)^{th}-$partial sum of the series $\sum_{kl} a_{kl}x_{kl}$ is in the space $\mathscr{BS}$ for these sequences $a = (a_{mn}) \in \Omega$ and $x = (x_{mn}) \in \widetilde{B}[\mathscr{C}_f]$ where $y = \widetilde{B}x \in [\mathscr{C}_f]$. By considering the similar way used in proving Theorem 0.9, we can summarize the rest of the proof as follows; we can say that $ax \in \mathscr{BS}$ whenever $x = (x_{mn}) \in \widetilde{B}[\mathscr{C}_f]$ if and only if $\widetilde{D}y \in \mathscr{M}_u$ whenever $y = (y_{mn}) \in [\mathscr{C}_f]$, where the matrix $\widetilde{D} = (\widetilde{d}_{mnkl})$ was defined by (20). This means that the conditions of Lemma 0.8 hold with the matrix $\widetilde{D} = (\widetilde{d}_{mnkl})$ instead of the matrix $A = (a_{mnkl})$. That is, $D_{mn} \in [\mathscr{C}_f]^{\beta(\vartheta)}$ for each fixed $m,n \in \mathbb{N}$ and

$$\sup_{m,n\in\mathbb{N}} \sum_{k,l} \left| \sum_{i,j=k,l}^{m,n} \frac{1}{r_i t_j} \prod_{\pi=k}^{i-1} \prod_{\rho=l}^{j-1} \left(\frac{-s_\pi}{r_\pi}\right) \left(\frac{-u_\rho}{t_\rho}\right) a_{ij} \right| < \infty.$$

Thus, the $\gamma-$dual $\{\widetilde{B}[\mathscr{C}_f]\}^{\gamma}$ of the space $\widetilde{B}[\mathscr{C}_f]$ is the set $\widetilde{d}_1 \cup CS_{\vartheta}$. This completes the proof. □

4. Matrix Transformations on the New Sequence Spaces $\widetilde{B}(\mathscr{C}_f)$ and $\widetilde{B}[\mathscr{C}_f]$

In this section, first we summarize the literature concerning the matrix transformations from and into the sequence spaces $\mathscr{C}_f$ and $[\mathscr{C}_f]$. Then we state some corollaries, without their proofs, which include characterization of some new four-dimensional matrix classes. We conclude the section with some significant results after stating the matrix classes $(\widetilde{B}(\mu) : \lambda)$ and $(\lambda : \widetilde{B}(\mu))$ which characterized by Tuğ et al. (2020) in general form.

Lemma 0.11. *(Zeltser, Mursaleen, & Mohiuddine, 2009, Theorem 3.1., p. 5) The following statements hold:*

(a) A four-dimensional matrix $A = (a_{mnkl})$ is almost $\mathscr{C}_{bp}-$conservative, i.e., $A \in (\mathscr{C}_{bp} : \mathscr{C}_f)$ if and only

if the condition in (10), and the following conditions hold

$$\exists a_{ij} \in \mathbb{C} \ni bp-\lim_{q,q'\to\infty} a(i,j,q,q',m,n) = a_{ij}, \tag{20}$$
uniformly in $m,n \in \mathbb{N}$ *for each* $i,j \in \mathbb{N}$

$$\exists u \in \mathbb{C} \ni bp-\lim_{q,q'\to\infty} \sum_{i,j} a(i,j,q,q',m,n) = u, \tag{21}$$
uniformly in $m,n \in \mathbb{N}$

$$\exists a_{ij} \in \mathbb{C} \ni bp-\lim_{q,q'\to\infty} \sum_{i} |a(i,j,q,q',m,n) - a_{ij}| = 0, \tag{22}$$
uniformly in $m,n \in \mathbb{N}$ *for each* $j \in \mathbb{N}$

$$\exists a_{ij} \in \mathbb{C} \ni bp-\lim_{q,q'\to\infty} \sum_{j} |a(i,j,q,q',m,n) - a_{ij}| = 0, \tag{23}$$
uniformly in $m,n \in \mathbb{N}$ *for each* $i \in \mathbb{N}$

where $a(i,j,q,q',m,n) = \sum_{k=m}^{m+q}\sum_{l=n}^{n+q'} a_{klij}/[(q+1)(q'+1)]$. *In this case,* $a = (a_{ij}) \in \mathscr{L}_u$ *and*

$$f_2 - \lim Ax = \sum_{i,j} a_{ij}x_{ij} + \left(u - \sum_{i,j} a_{ij}\right) bp - \lim_{i,j\to\infty} x_{ij},$$

that is,

$$bp-\lim_{q,q'\to\infty} \sum_{i,j} a(i,j,q,q',m,n)x_{ij} = \sum_{i,j} a_{ij}x_{ij} + \left(u - \sum_{i,j} a_{ij}\right) bp - \lim_{i,j\to\infty} x_{ij},$$
uniformly in $m,n \in \mathbb{N}$.

(b) *A four-dimensional matrix* $A = (a_{mnkl})$ *is almost* $\mathscr{C}_{bp}-$*regular, i.e.,* $A \in (\mathscr{C}_{bp} : \mathscr{C}_f)_{reg}$ *if and only if the conditions (10), (20)-(23) hold with* $a_{ij} = 0$ *for all* $i,j \in \mathbb{N}$ *and* $u = 1$

Lemma 0.12. *(Zeltser et al., 2009, Theorem 3.2., p.9) The following statements hold:*

(a) *A four-dimensional matrix* $A = (a_{mnkl})$ *is almost* $\mathscr{C}_r-$*conservative, i.e.,* $A \in (\mathscr{C}_r : \mathscr{C}_f)$ *if and only if the conditions in (10), (20) and (21), and the following conditions hold*

$$\exists j_0 \in \mathbb{N} \ni bp-\lim_{q,q'\to\infty} \sum_{i} a(i,j_0,q,q',m,n) = u_{j_0}, \tag{24}$$
uniformly in $m,n \in \mathbb{N}$,
$$\exists i_0 \in \mathbb{N} \ni bp-\lim_{q,q'\to\infty} \sum_{j} a(i_0,j,q,q',m,n) = v_{i_0}, \tag{25}$$
uniformly in $m,n \in \mathbb{N}$,

where $a(i,j,q,q',m,n)$ *is defined as in the Lemma 0.11. In this case,* $a = (a_{ij}) \in \mathscr{L}_u$; $(u_j),(v_i) \in \ell_1$ *and*

$$\begin{aligned} f_2 - \lim Ax &= \sum_{ij} a_{ij}x_{ij} + \sum_i \left(v_i - \sum_j a_{ij}\right) x_i + \sum_j \left(u_j - \sum_i a_{ij}\right) x_j \\ &+ \left(u + \sum_{i,j} a_{ij} - \sum_i v_i - \sum_j u_j\right) r - \lim_{i,j\to\infty} x_{ij}. \end{aligned}$$

(b) A four-dimensional matrix $A = (a_{mnkl})$ is almost $\mathscr{C}_r-$regular, i.e., $A \in (\mathscr{C}_r : \mathscr{C}_f)_{reg}$ if and only if the conditions in (10), (20), (21), (25) and (25) hold with $a_{ij} = u_j = v_i = 0$ for all $i, j \in \mathbb{N}$ and $u = 1$.

Lemma 0.13. *(Zeltser et al., 2009, Theorem 3.3., p.11) The following statements hold:*

(a) A four-dimensional matrix $A = (a_{mnkl})$ is almost $\mathscr{C}_p-$conservative, i.e., $A \in (\mathscr{C}_p : \mathscr{C}_f)$ if and only if the conditions in (10), (20) and (21) hold, and

$$\forall k \in \mathbb{N},\ \exists K \in \mathbb{N} \ni a_{mnkl} = 0 \ for\ l > K,\ (m,n \in \mathbb{N}), \tag{26}$$

$$\forall l \in \mathbb{N},\ \exists L \in \mathbb{N} \ni a_{mnkl} = 0 \ for\ k > L,\ (m,n \in \mathbb{N}). \tag{27}$$

In this case $a = (a_{ij}) \in \mathscr{L}_u, (a_{ij_0})_{i\in\mathbb{N}}, (a_{i_0 j})_{j\in\mathbb{N}} \in \varphi$ where φ denotes the space of all finitely non-zero sequences and

$$f_2 - \lim Ax = \sum_{i,j} a_{ij}x_{ij} + \left(u - \sum_{i,j} a_{ij}\right) p - \lim_{i,j\to\infty} x_{ij}.$$

(b) A four-dimensional matrix $A = (a_{mnkl})$ is almost $\mathscr{C}_p-$regular, i.e., $A \in (\mathscr{C}_r : \mathscr{C}_f)_{reg}$ if and only if the conditions in (10), (20), (21),(26) and (27) hold with $a_{ij} = 0$ for all $i, j \in \mathbb{N}$ and $u = 1$.

Lemma 0.14. *(Mursaleen, 2004, Theorem 2.2., p.527) A four-dimensional matrix $A = (a_{mnkl})$ is almost strongly regular, i.e., $A \in (\mathscr{C}_f : \mathscr{C}_f)_{reg}$ if and only if A is almost regular and the following two conditions hold*

$$\lim_{q,q'\to\infty} \sum_i \sum_j \left|\Delta_{10} a(i,j,q,q',m,n)\right| = 0 \ uniformly\ in\ m,n \in \mathbb{N}, \tag{28}$$

$$\lim_{q,q'\to\infty} \sum_j \sum_i \left|\Delta_{01} a(i,j,q,q',m,n)\right| = 0 \ uniformly\ in\ m,n \in \mathbb{N}, \tag{29}$$

where

$$\Delta_{10} a(i,j,q,q',m,n) = a(i,j,q,q',m,n) - a(i+1,j,q,q',m,n),$$
$$\Delta_{01} a(i,j,q,q',m,n) = a(i,j,q,q',m,n) - a(i,j+1,q,q',m,n).$$

Lemma 0.15. *(Yeşilkayagil & Başar, 2016, Theorem 3.5., p,43) The four-dimensional matrix $A = (a_{mnkl}) \in (\mathscr{M}_u : \mathscr{C}_f)$ if and only if the condition (10) and the following conditions hold*

$$\exists \beta_{kl} \in \mathbb{C} \ni f_2 - \lim_{m,n\to\infty} a_{mnkl} = \beta_{kl} \ for\ all\ k,l \in \mathbb{N}, \tag{30}$$

$$For\ every\ m,n,j \in \mathbb{N}, \exists K \in \mathbb{N} \ni \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} a_{klij} = 0, \tag{31}$$
$$for\ all\ q,q',i > K,$$

$$For\ every\ m,n,i \in \mathbb{N}, \exists L \in \mathbb{N} \ni \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} a_{klij} = 0, \tag{32}$$
$$for\ all\ q,q',j > L.$$

Lemma 0.16. *(Tug, 2018, Theorem 3., p.4) Let $A = (a_{mnkl})$ be a four-dimensional infinite matrix. Then the following statements hold.*

(a) Let $0 < s' \leq 1$. Then, $A = (a_{mnkl}) \in (\mathscr{L}_{s'} : \mathscr{C}_f)$ if and only if

$$\sup_{m,n,k,l\in\mathbb{N}} |a_{mnkl}| < \infty, \tag{33}$$

$$\exists (a_{kl}) \in \mathbb{C} \ such\ that\ f_2 - \lim_{m,n\to\infty} a_{mnkl} = a_{kl} \ for\ all\ k,l \in \mathbb{N} \tag{34}$$

(b) *Let* $1 < s' < \infty$. *Then,* $A = (a_{mnkl}) \in (\mathscr{L}_{s'} : \mathscr{C}_f)$ *if and only if the condition (34) holds and*

$$\sup_{m,n\in\mathbb{N}} \sum_{k,l} |a_{mnkl}|^{s'} < \infty. \tag{35}$$

Lemma 0.17. *(Tuğ, 2021, Theorem 4.2., p.13) Four-dimensional matrix* $A = (a_{mnkl}) \in ([\mathscr{C}_f] : \mathscr{C}_f)$ *with* $f_2 - \lim Ax = [f_2] - \lim_{kl} x_{kl}$ *if and only if A is almost* $\mathscr{C}_{bp}-$*regular,i.e.,* $A = (a_{mnkl}) \in (\mathscr{C}_{bp} : \mathscr{C}_f)$ *with* $f_2 - \lim Ax = bp - \lim_{kl} x_{kl}$ *and*

$$\sum_{k,l\in E} |\Delta_{11} a_{mnkl}| \to 0, \ \ as\ m,n \to \infty \tag{36}$$

for each set E which is uniformly zero density where

$$\Delta_{11} a_{mnkl} = a_{mnkl} - a_{mn,k+1,l} - a_{mn,k,l+1} + a_{mn,k+1,l+1} \tag{37}$$

Now we come up with the following corollaries without their proofs.

Corollary 0.18. *The four-dimensional matrix* $A = (a_{mnkl}) \in (\mathscr{M}_u : [\mathscr{C}_f])$ *if and only if the condition (10) and the following conditions hold*

$$\exists \beta_{kl} \in \mathbb{C} \ni [f_2] - \lim_{m,n\to\infty} a_{mnkl} = \beta_{kl}\ for\ all\ k,l \in \mathbb{N}, \tag{38}$$

$$For\ every\ m,n,j \in \mathbb{N}, \exists K \in \mathbb{N} \ni \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} |a_{klij}| = 0, \tag{39}$$
$$for\ all\ q,q',i > K,$$

$$For\ every\ m,n,i \in \mathbb{N}, \exists L \in \mathbb{N} \ni \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} |a_{klij}| = 0, \tag{40}$$
$$for\ all\ q,q',j > L.$$

Corollary 0.19. *Let* $A = (a_{mnkl})$ *be a four-dimensional infinite matrix. Then the following statements hold.*

(a) *Let* $0 < s' \leq 1$. *Then,* $A = (a_{mnkl}) \in (\mathscr{L}_{s'} : [\mathscr{C}_f])$ *if and only if (33) holds and*

$$\exists (a_{kl}) \in \mathbb{C}\ such\ that\ [f_2] - \lim_{m,n\to\infty} a_{mnkl} = a_{kl}\ for\ all\ k,l \in \mathbb{N} \tag{41}$$

(b) *Let* $1 < s' < \infty$. *Then,* $A = (a_{mnkl}) \in (\mathscr{L}_{s'} : [\mathscr{C}_f])$ *if and only if the condition (41) holds and*

$$\sup_{m,n\in\mathbb{N}} \sum_{k,l} |a_{mnkl}|^{s'} < \infty. \tag{42}$$

Corollary 0.20. *The following statements hold:*

(a) *A four-dimensional matrix* $A = (a_{mnkl})$ *is strongly almost* $\mathscr{C}_{bp}-$*conservative, i.e.,* $A \in (\mathscr{C}_{bp} : [\mathscr{C}_f])$ *if* *and only if the condition in (10), and the following conditions hold*

$$\exists a_{ij} \in \mathbb{C} \ni bp - \lim_{q,q' \to \infty} \widetilde{a}(i,j,q,q',m,n) = a_{ij}, \tag{43}$$
uniformly in $m,n \in \mathbb{N}$ *for each* $i,j \in \mathbb{N}$
$$\exists u \in \mathbb{C} \ni bp - \lim_{q,q' \to \infty} \sum_{i,j} \widetilde{a}(i,j,q,q',m,n) = u, \tag{44}$$
uniformly in $m,n \in \mathbb{N}$

$$\exists a_{ij} \in \mathbb{C} \ni bp - \lim_{q,q' \to \infty} \sum_{i} |\widetilde{a}(i,j,q,q',m,n) - a_{ij}| = 0, \tag{45}$$
uniformly in $m,n \in \mathbb{N}$ *for each* $j \in \mathbb{N}$

$$\exists a_{ij} \in \mathbb{C} \ni bp - \lim_{q,q' \to \infty} \sum_{j} |\widetilde{a}(i,j,q,q',m,n) - a_{ij}| = 0, \tag{46}$$
uniformly in $m,n \in \mathbb{N}$ *for each* $i \in \mathbb{N}$

where $\widetilde{a}(i,j,q,q',m,n) = \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} |a_{klij}|$. *In this case,* $a = (a_{ij}) \in \mathscr{L}_u$ *and*

$$[f_2] - \lim Ax = \sum_{i,j} a_{ij}x_{ij} + \left(u - \sum_{i,j} a_{ij} \right) bp - \lim_{i,j \to \infty} x_{ij},$$

that is,

$$bp - \lim_{q,q' \to \infty} \sum_{i,j} \widetilde{a}(i,j,q,q',m,n)x_{ij} = \sum_{i,j} a_{ij}x_{ij} + \left(u - \sum_{i,j} a_{ij} \right) bp - \lim_{i,j \to \infty} x_{ij},$$
uniformly in $m,n \in \mathbb{N}$.

(b) *A four-dimensional matrix* $A = (a_{mnkl})$ *is strongly almost* $\mathscr{C}_{bp}-$*regular, i.e.,* $A \in (\mathscr{C}_{bp} : [\mathscr{C}_f])_{reg}$ *if and only if the conditions (10), (43)-(46) hold with* $a_{ij} = 0$ *for all* $i,j \in \mathbb{N}$ *and* $u = 1$.

Corollary 0.21. *Four-dimensioanl matrix* $A = (a_{mnkl}) \in ([\mathscr{C}_f] : [\mathscr{C}_f])$ *with* $[f_2] - \lim Ax = [f_2] - \lim_{kl} x_{kl}$ *if and only if A is strongly almost* $\mathscr{C}_{bp}-$*regular,i.e.,* $A = (a_{mnkl}) \in (\mathscr{C}_{bp} : [\mathscr{C}_f])$ *with* $[f_2] - \lim Ax = bp - \lim_{kl} x_{kl}$ *and the condition in (36) holds.*

Corollary 0.22. *A four-dimensional matrix* $A = (a_{mnkl})$ *is strongly almost strongly regular, i.e.,* $A \in (\mathscr{C}_f : [\mathscr{C}_f])_{reg}$ *if and only if A is strongly almost regular and the following two conditions hold*

$$\lim_{q,q' \to \infty} \sum_i \sum_j \left| \Delta_{10}\widetilde{a}(i,j,q,q',m,n) \right| = 0 \text{ uniformly in } m,n \in \mathbb{N}, \tag{47}$$
$$\lim_{q,q' \to \infty} \sum_j \sum_i \left| \Delta_{01}\widetilde{a}(i,j,q,q',m,n) \right| = 0 \text{ uniformly in } m,n \in \mathbb{N}, \tag{48}$$

where $\widetilde{a}(i,j,q,q',m,n) = \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} |a_{klij}|$ *and*

$$\Delta_{10}\widetilde{a}(i,j,q,q',m,n) = \widetilde{a}(i,j,q,q',m,n) - \widetilde{a}(i+1,j,q,q',m,n),$$
$$\Delta_{01}\widetilde{a}(i,j,q,q',m,n) = \widetilde{a}(i,j,q,q',m,n) - \widetilde{a}(i,j+1,q,q',m,n).$$

Now, the following Lemma 0.23 is given by considering four-dimensional dual summability method for double sequences.

Lemma 0.23. *(Tuğ et al., 2020, Theorem 13., p.14) Suppose that the four-dimensional matrices $A = (a_{mnkl})$ and $\widetilde{C} = (\widetilde{c}_{mnkl})$ are connected with the following relation*

$$\widetilde{c}_{mnkl} = \sum_{i,j=k,l}^{\infty} \frac{1}{r_i t_j} \prod_{\pi=k}^{i-1} \prod_{\rho=l}^{j-1} \left(\frac{-s_\pi}{r_\pi}\right) \left(\frac{-u_\rho}{t_\rho}\right) a_{mnij} \tag{49}$$

for any $m,n,k,l \in \mathbb{N}$ and λ is any given double sequence space. Then, $A \in (\widetilde{B}(\mu) : \lambda)$ if and only if

$$A_{mn} \in \{\widetilde{B}(\mu)\}^{\beta(\vartheta)} \text{ for any } m,n \in \mathbb{N}, \tag{50}$$

$$\widetilde{C} \in (\mu : \lambda) \tag{51}$$

Corollary 0.24. *Suppose that the four-dimensional matrices $A = (a_{mnkl})$ and $\widetilde{C} = (\widetilde{c}_{mnkl})$ are connected with the following relation (49). The the followings hold;*

(a) $A \in (\widetilde{B}(\mathscr{C}_{bp}) : \mathscr{C}_f)$ *if and only if the conditions in (50) holds, and (10), (20)-(23) hold with $\widetilde{c}_{mnkl}$ instead of a_{mnkl}.*

(b) $A \in (\widetilde{B}(\mathscr{C}_r) : \mathscr{C}_f)$ *if and only if the conditions in (50) holds, and (10), (20), (21), (24), (25) hold with $\widetilde{c}_{mnkl}$ instead of a_{mnkl}.*

(c) $A \in (\widetilde{B}(\mathscr{C}_p) : \mathscr{C}_f)$ *if and only if the conditions in (50) holds, and (10), (20), (21), (26), (27) hold with $\widetilde{c}_{mnkl}$ instead of a_{mnkl}.*

(d) $A \in (\widetilde{B}(\mathscr{L}_s) : \mathscr{C}_f)$, $(0 < s' \leq 1)$ *if and only if the conditions in (50) holds, and (33) and (34) hold with $\widetilde{c}_{mnkl}$ instead of a_{mnkl}.*

(e) $A \in (\widetilde{B}(\mathscr{L}_s) : \mathscr{C}_f)$, $(1 < s' < \infty)$ *if and only if the conditions in (50) holds, and (34) and (35) hold with $\widetilde{c}_{mnkl}$ instead of a_{mnkl}.*

(f) $A \in (\widetilde{B}(\mathscr{C}_f) : \mathscr{C}_f)$ *if and only if the conditions in (50) holds, and (10), (20)-(23) hold with $a_{ij} = 0$ for all $i,j \in \mathbb{N}$ and $u = 1$, and (28) and (29) hold with $\widetilde{c}_{mnkl}$ instead of a_{mnkl}.*

Corollary 0.25. *Suppose that the four-dimensional matrices $A = (a_{mnkl})$ and $\widetilde{C} = (\widetilde{c}_{mnkl})$ are connected with the following relation (49). The the followings hold;*

(a) $A \in (\widetilde{B}(\mathscr{C}_{bp}) : [\mathscr{C}_f])$ *if and only if the conditions in (50) holds, and (10), (43)-(46) hold with $\widetilde{c}_{mnkl}$ instead of a_{mnkl}.*

(b) $A \in (\widetilde{B}(\mathscr{L}_s) : [\mathscr{C}_f])$, $(0 < s' \leq 1)$ *if and only if the conditions in (50) holds, and (33) and (41) hold with $\widetilde{c}_{mnkl}$ instead of a_{mnkl}.*

(c) $A \in (\widetilde{B}(\mathscr{L}_s) : [\mathscr{C}_f])$, $(1 < s' < \infty)$ *if and only if the conditions in (50) holds, and (41) and (42) hold with $\widetilde{c}_{mnkl}$ instead of a_{mnkl}.*

(d) $A \in (\widetilde{B}(\mathscr{C}_f) : [\mathscr{C}_f])$ *if and only if the conditions in (50) holds, and (10), (43)-(46) hold with $a_{ij} = 0$ for all $i,j \in \mathbb{N}$ and $u = 1$, and (30)-(32) hold with $\widetilde{c}_{mnkl}$ instead of a_{mnkl}.*

(e) $A \in (\widetilde{B}(\mathscr{M}_u) : [\mathscr{C}_f])$ *if and only if the conditions in (50) holds, and (10), (38)-(40) hold with $\widetilde{c}_{mnkl}$ instead of a_{mnkl}.*

(f) $A = (a_{mnkl}) \in (\widetilde{B}[\mathscr{C}_f] : [\mathscr{C}_f])$ *with* $[f_2] - \lim Ax = [f_2] - \lim_{kl} x_{kl}$ *if and only if* $A = (a_{mnkl}) \in (\widetilde{B}(\mathscr{C}_{bp}) : [\mathscr{C}_f])$ *with* $[f_2] - \lim Ax = bp - \lim_{kl} x_{kl}$ *and the condition in (36) holds with $\widetilde{c}_{mnkl}$ instead of a_{mnkl}.*

Corollary 0.26. *Suppose that the four-dimensional matrices $A = (a_{mnkl})$ and $\widetilde{C} = (\widetilde{c}_{mnkl})$ are connected with the following relation (49). The the followings hold;*

(a) $A \in (\widetilde{B}(\mathscr{C}_f) : \mathscr{C}_{bp})$ *if and only if the conditions in (50) holds, and (10)-(16) hold with $\widetilde{c}_{mnkl}$ instead of a_{mnkl}.*

(b) $A \in (\widetilde{B}(\mathscr{C}_f) : \mathscr{M}_u)$ *if and only if the conditions in (50) holds, and (10) holds with $\widetilde{c}_{mnkl}$ instead of a_{mnkl}.*

(c) $A \in (\widetilde{B}[\mathscr{C}_f] : \mathscr{C}_f)$ *if and only if the conditions in (50) holds, and (20)-(23) and (36) hold with $\widetilde{c}_{mnkl}$ instead of a_{mnkl}.*

(d) $A \in (\widetilde{B}[\mathscr{C}_f] : \mathscr{C}_{bp})$ *if and only if the conditions in (50) holds, and (11)-(14) hold with $a_{ij} = 0$ for all $i, j \in \mathbb{N}$ and $u = 1$, and (18)-(19) hold with $\widetilde{c}_{mnkl}$ instead of a_{mnkl}.*

(e) $A \in (\widetilde{B}[\mathscr{C}_f] : \mathscr{M}_u)$ *if and only if the conditions in (50) holds, and (10) holds with $\widetilde{c}_{mnkl}$ instead of a_{mnkl}.*

Lemma 0.27. *(Tuğ et al., 2020, Theorem 14., p.15) Suppose that the four-dimensional infinite matrices $A = (a_{mnkl})$ and $\widetilde{G} = (\widetilde{g}_{mnkl})$ are connected with the following relation*

$$\begin{aligned} \widetilde{g}_{mnkl} &= \sum_{i,j=0}^{m,n} \widetilde{b}_{mnij} a_{ijkl} \\ &= s_{m-1}u_{n-1}a_{m-1,n-1,k,l} + s_{m-1}t_n a_{m-1,n,k,l} + r_m u_{n-1} a_{m,n-1,k,l} + r_m t_n a_{mnkl} \end{aligned} \tag{52}$$

for all $m,n,k,l \in \mathbb{N}$. Then, $A \in (\lambda : \widetilde{B}(\mu))$ if and only if

$$\widetilde{G} \in (\lambda : \mu) \tag{53}$$

Corollary 0.28. *Suppose that the four-dimensional matrices $A = (a_{mnkl})$ and $\widetilde{G} = (\widetilde{g}_{mnkl})$ are connected with the following relation (49). The the followings hold;*

(a) $A \in (\mathscr{C}_{bp} : \widetilde{B}(\mathscr{C}_f))$ *if and only if the conditions in (10), (20)-(23) hold with $\widetilde{g}_{mnkl}$ instead of a_{mnkl}.*

(b) $A \in (\mathscr{C}_r : \widetilde{B}(\mathscr{C}_f))$ *if and only if the conditions in (10), (20), (21), (24), (25) hold with $\widetilde{g}_{mnkl}$ instead of a_{mnkl}.*

(c) $A \in (\mathscr{C}_p : \widetilde{B}(\mathscr{C}_f))$ *if and only if the conditions in (10), (20), (21), (26), (27) hold with $\widetilde{g}_{mnkl}$ instead of a_{mnkl}.*

(d) $A \in (\mathscr{L}_s : \widetilde{B}(\mathscr{C}_f))$, $(0 < s' \leq 1)$ *if and only if the conditions in (33) and (34) hold with $\widetilde{g}_{mnkl}$ instead of a_{mnkl}.*

(e) $A \in (\mathscr{L}_s : \widetilde{B}(\mathscr{C}_f))$, $(1 < s' < \infty)$ *if and only if the conditions in (34) and (35) hold with $\widetilde{g}_{mnkl}$ instead of a_{mnkl}.*

(f) $A \in (\mathscr{C}_f : \widetilde{B}(\mathscr{C}_f))$ *if and only if the conditions in (10), (20)-(23) hold with $a_{ij} = 0$ for all $i, j \in \mathbb{N}$ and $u = 1$, and (28) and (29) hold with $\widetilde{g}_{mnkl}$ instead of a_{mnkl}.*

Corollary 0.29. *Suppose that the four-dimensional matrices $A = (a_{mnkl})$ and $\widetilde{G} = (\widetilde{g}_{mnkl})$ are connected with the following relation (49). The the followings hold;*

(a) $A \in (\mathscr{C}_{bp} : \widetilde{B}[\mathscr{C}_f])$ *if and only if the conditions in (10), (43)-(46) hold with $\widetilde{g}_{mnkl}$ instead of a_{mnkl}.*

(b) $A \in (\mathscr{L}_s : \widetilde{B}[\mathscr{C}_f])$, $(0 < s' \leq 1)$ *if and only if the conditions in (33) and (41) hold with $\widetilde{g}_{mnkl}$ instead of a_{mnkl}.*

(c) $A \in (\mathscr{L}_s : \widetilde{B}[\mathscr{C}_f])$, $(1 < s' < \infty)$ *if and only if the conditions in (41) and (42) hold with* $\widetilde{g}_{mnkl}$ *instead of* a_{mnkl}.

(d) $A \in (\mathscr{C}_f : \widetilde{B}[\mathscr{C}_f])$ *if and only if the conditions in (10), (43)-(46) hold with* $a_{ij} = 0$ *for all* $i, j \in \mathbb{N}$ *and* $u = 1$, *and (30)-(32) hold with* $\widetilde{g}_{mnkl}$ *instead of* a_{mnkl}.

(e) $A \in (\mathscr{M}_u : \widetilde{B}[\mathscr{C}_f])$ *if and only if the conditions in (10), (38)-(40) hold with* $\widetilde{g}_{mnkl}$ *instead of* a_{mnkl}.

(f) $A = (a_{mnkl}) \in ([\mathscr{C}_f] : \widetilde{B}[\mathscr{C}_f])$ *with* $[f_2] - \lim Ax = [f_2] - \lim_{kl} x_{kl}$ *if and only if* $A = (a_{mnkl}) \in (\mathscr{C}_{bp} : \widetilde{B}[\mathscr{C}_f])$ *with* $[f_2] - \lim Ax = bp - \lim_{kl} x_{kl}$ *and the condition in (36) holds with* $\widetilde{g}_{mnkl}$ *instead of* a_{mnkl}.

Corollary 0.30. *Suppose that the four-dimensional matrices* $A = (a_{mnkl})$ *and* $\widetilde{G} = (\widetilde{g}_{mnkl})$ *are connected with the following relation (49). The the followings hold;*

(a) $A \in (\mathscr{C}_f : \widetilde{B}(\mathscr{C}_{bp}))$ *if and only if (10)-(16) hold with* $\widetilde{g}_{mnkl}$ *instead of* a_{mnkl}.

(b) $A \in (\mathscr{C}_f : \widetilde{B}(\mathscr{M}_u))$ *if and only if (10) holds with* $\widetilde{g}_{mnkl}$ *instead of* a_{mnkl}.

(c) $A \in ([\mathscr{C}_f] : \widetilde{B}(\mathscr{C}_f))$ *if and only if (20)-(23) and (36) hold with* $\widetilde{g}_{mnkl}$ *instead of* a_{mnkl}.

(d) $A \in ([\mathscr{C}_f] : \widetilde{B}(\mathscr{C}_{bp}))$ *if and only if (11)-(14) hold with* $a_{ij} = 0$ *for all* $i, j \in \mathbb{N}$ *and* $u = 1$, *and (18)-(19) hold with* $\widetilde{g}_{mnkl}$ *instead of* a_{mnkl}.

(e) $A \in ([\mathscr{C}_f] : \widetilde{B}(\mathscr{M}_u))$ *if and only if (10) holds with* $\widetilde{g}_{mnkl}$ *instead of* a_{mnkl}.

5. Conclusion

Lorentz (1948) introduced almost convergence for single sequence and then M. Mursaleen (2010) investigated the certain properties of the space of almost convergent sequences denoted by f. Then many of the mathematician has studied the matrix domain on almost null and almost convergent sequences spaces (see Başar and Kirişçi (2011), Tuğ and Başar (2016), Kayaduman and Şengönül (2012), Şengönül and Kayaduman (2012).RhoadesMoricz and Rhoades (1988) introduced and studied almost convergence for double sequences. Then many significant contributions have been done by several mathematicians (see Tuğ (2018), Tug (2018), Tuğ (2021), Móricz and Rhoades (1990), Mursaleen and Savaş (2003), Cunjalo (2007), M. Mursaleen and Mohiuddine (2009), M. Mursaleen and Mohiuddine (2010).In this paper, we defined the spaces $\widetilde{B}(\mathscr{C}_f)$, $\widetilde{B}(\mathscr{C}_{f_0})$, $\widetilde{B}[\mathscr{C}_f]$ and $\widetilde{B}[\mathscr{C}_{f_0}]$, where the matrix $B(\widetilde{r}, \widetilde{s}, \widetilde{t}, \widetilde{u})$ was defined by Tuğ et al. (2020) as four-dimensional sequential band matrix. Then we state some topological properties beside some strict inclusion relations. In the third section, we calculated the $\alpha-, \beta bp-$ and $\gamma-$dual spaces of $\widetilde{B}(\mathscr{C}_f)$ and $\widetilde{B}[\mathscr{C}_f]$. In the last section, we stated the known matrix classes from or into the spaces $\widetilde{B}(\mathscr{C}_f)$ and $\widetilde{B}[\mathscr{C}_f]$, and then we characterized some new matrix classes $(\widetilde{B}(\mu) : \lambda)$ and $(\lambda : \widetilde{B}(\mu))$, where λ, μ are any sequence spaces from the set $\{\mathscr{C}_f, [\mathscr{C}_f], \mathscr{C}_{bp}, \mathscr{C}_r, \mathscr{C}_p, \mathscr{L}_{s'}, \mathscr{M}_u\}$.

References

Adams, C. R. (1933). On non-factorable transformations of double sequences. *Proc. Natl. Acad. Sci. USA*, *19*(5), 564.

Başar, F., & Kirişçi, M. (2011). Almost convergence and generalized difference matrix. *Comput. Math. Appl.*, *61*(3), 602–611.

Başarir, M. (1995). On the strong almost convergence of double sequences. *Period. Math. Hungar*, *30*(3), 177–181.

Çapan, H., & Başar, F. (2019). On the difference spaces of almost convergent and strongly almost convergent double sequences. *Positivity*, *23*(2), 493–506.

Cunjalo, F. (2007). Almost convergence of double sequences-some analogies between measure and category. *Math. Maced*, *5*, 21–24.

Kayaduman, K., & Şengönül, M. (2012). The spaces of cesaro almost convergent sequences and core theorems. *Acta Math. Sci. Ser. B*, *32*(6), 2265–2278.

Lorentz, G. G. (1948). A contribution to the theory of divergent sequences. *Acta math.*, *80*(1), 167–190.

Moricz, F., & Rhoades, B. (1988). Almost convergence of double sequences and strong regularity of summability matrices. In *Math. proc. cambridge philos. soc.* (Vol. 104, pp. 283–294).

Móricz, F., & Rhoades, B. (1990). Some characterizations of almost convergence for single and double sequences. In *Proceedings of the 3rd annual meeting of the international workshop in analysis and its applications* (Vol. 48, pp. 61–68).

Mursaleen. (2004). Almost strongly regular matrices and a core theorem for double sequences. *J. Math. Anal. Appl.*, *293*(2), 523–531.

Mursaleen, & Savaş, E. (2003). Almost regular matrices for double sequences. *Studia Sci. Math. Hungar.*, *40*(1-2), 205–212.

Mursaleen, M. (2010). Almost convergence and some related methods. *Modern Methods of Analysis and Its Applications*, 1–10.

Mursaleen, M., & Mohiuddine, S. (2009). Almost bounded variation of double sequences and some four dimensional summability matrices. *Publ. Math. Debrecen*, *75*(3-4), 495–508.

Mursaleen, M., & Mohiuddine, S. (2010). Invariant mean and some core theorems for double sequences. *Taiwanese J. Math.*, *14*(1), 21–33.

Robison, G. M. (1926). Divergent double sequences and series. *Trans. Amer. Math. Soc.*, *28*(1), 50–73.

Şengönül, M., & Kayaduman, K. (2012). On the riesz almost convergent sequences space. In *Abstr. appl. anal.* (Vol. 2012).

Tuğ, O. (2017a). Four-dimensional generalized difference matrix and almost convergent double sequence spaces. In *Symposium functional analysis in interdisciplinary applications* (pp. 83–87).

Tuğ, O. (2017b). Four-dimensional generalized difference matrix and some double sequence spaces. *J. Inequal. Appl.*, *2017*(1), 1–22.

Tuğ, O. (2021). The spaces of$b(r,s,t,u)$ strongly almost convergent double sequences and matrix transformations. *Bull. Sci. Math.*, *169*, 102989.

Tuğ, O., & Başar, F. (2016). On the spaces of nörlund almost null and nörlund almost convergent sequences. *Filomat*, *30*(3), 773–783.

Tuğ, O. (2017). On the matrix domain in the sequence space $\mathscr{L}_u$. *Eurasian Journal of Science and Engineering*, *3*(1), 204–211.

Tuğ, O. (2018). On almost b-summable double sequence spaces. *J. Inequal. Appl.*, *2018*(1), 1–19.

Tug, O. (2018). On the characterization of some classes of four-dimensional matrices and almost-summable double sequences. *Journal of Mathematics*, *2018*.

Tuğ, O., & Başar, F. (2016). Four-dimensional generalized difference matrix and some double sequence spaces. In *Aip conference proceedings* (Vol. 1759, p. 020075).

Tuğ, O., Rakočević, V., & Malkowsky, E. (2020). On the domain of the four-dimensional sequential band matrix in some double sequence spaces. *Mathematics*, *8*(5), 789.

Yeşilkayagil, M., & Başar, F. (2016). On the characterization of a class of four dimensional matrices and steinhaus type theorems. *Kragujevac J. Math.*, *40*(1), 35–45.

Zeltser, M., Mursaleen, M., & Mohiuddine, S. (2009). On almost conservative matrix methods for double sequence spaces. *Publ. Math. Debrecen*, *75*(3–4), 387–399.