Hybridization of Genetic Algorithm with Homotopy Analysis Method for Solving Fractional Partial Differential Equations

Ahmed Entesar[1] & Omar Saber[2] & Waleed Al-Hayani[3]

[1,2,3] Department of Mathematics, College of Computer Science and Mathematics, University of Mosul, Iraq
Correspondence: Omar Saber, University of Mosul, Iraq.
E-mail address: omarsaber79.os@gmail.com.

Received: October 10, 2018 Accepted: November 23, 2018 Online Published: December 1, 2019

doi: 10.23918/eajse.v4i2p19

Abstract: In this work, the Homotopy Analysis Method (HAM) is applied to solve fractional Partial Differential Equations (PDEs). The solution of HAM has improved the results by using Genetic Algorithm (GA). The hybrid method (proposed) is applied for types of problems where analytical solutions approximate are obtained. Numerical experiments are also presented.

Keywords: Homotopy Analysis Method (HAM), Genetic Algorithm (GA), Heat Like Equations, Wave Like Equations, Fractional Calculus

1. Introduction

Such equations are widely used in the fields of physics, chemistry, engineering and the rest of the other sciences. They are the equations of derivation and fractional integration, and some theories of derivatives of fractional integration with the incorrect number (Kilbas & Srivastava, & Trujillo, 2006; Podlubny, 1999; Luchko & Groreflo, 1998). A survey of some implementation of fractional calculus in continuum and statistical techniques is given by Mainardi (1997). Oldham and Spanier (1974) and Ross and Miller (1993) provide the history and a comprehensive treatment of this topic. Through these past years, this idea was developed by Leibniz to L'Hôpital in 1695, which used integration and fractional derivation in several fields, including physical engineering, engineering and other sciences, and provided many definitions and hypotheses for solving such fractional equations

When the time period is large, fractional derivatives are generally made up of micro-generators. The importance of research in these equations is the need to develop the concept of equilibrium, and also for the stability of these equations (Fujita, 1990; Hilfer, 1995, 2000).

In this paper, we will consider the fractional heat-like and wave-like equations of the form.

$$\frac{\partial^\alpha u}{\partial t^\alpha} = f(x,y,z)u_{xx} + g(x,y,z)u_{yy} + f(x,y,z)u_{zz}, \quad (1)$$
$$0 < x < a, 0 < y < b, 0 < z < c, 0 < \alpha \le 2, t > 0,$$

subject to the Neumann boundary conditions

$$\begin{aligned} &u(0,y,z,t) = f_1(y,z,t), \qquad &&u_x(a,y,z,t) = f_2(y,z,t),\\ &u(x,0,z,t) = g_1(x,z,t), \qquad &&u_y(x,b,z,t) = g_2(x,z,t),\\ &u(x,y,0,t) = h_1(x,y,t), \qquad &&u_z(x,y,c,t) = h_2(x,y,t), \end{aligned} \tag{2}$$

and the initial conditions

$$u(x,y,z,0) = \psi(x,y,z), \qquad u_t(x,y,z,0) = \phi(x,y,z), \tag{3}$$

Whereu_t is the rate of change of temperature at a point over time and α is a parameter describing the fractional derivative and $u = u(x,y,z,t)$ is temperature as a function of time and space, while u_{xx}, u_{yy} and u_{zz} are the second spatial derivatives of temperature in x, y, and z directions, respectively. Finally, $f(x,y,z)$, $g(x,y,z)$ and $h(x,y,z)$ are any functions with respect to$x$, y, and z.

When $0 < \alpha \leq 1$, equation (1) reduces to a fractional heat-like equation with variable coefficients and when$1 < \alpha \leq 2$ equation (1) reduces to a fractional wave-like equation which models anomalous diffusive and subdiffusive systems, description of fractional random walk, unification of diffusion and wave propagation phenomena (Agrawal, 2002, Andrezei, 2002, Metzler & Klafter, 2000). Recently, Molliq et al. (2009) applied the variational iteration method for solving equation (1). Momani (2005) and Al-Hayani (2017) solved the same equation (use the Adomian decomposition method (ADM).

The HAMcan solve all types of linear, nonlinear, partial, normal and fractional equations also (Duld, 1997) and give a close series of the exact solution, using the Maclaurin series. This method was first discovered by Liao Shijun in 1992 in a doctoral dissertation from the University of Shanghai and was also developed (Liao, 1997, 1999) by adding a non-zero parameter. Through this parameter we can control the solution and obtain a series close to the exact solution. The advantage of this method is solving nonlinear equations without invoking unrealistic assumptions, discretization or linearization.

The primary objective of our research is to apply HAM to have fractional solutions for (1).VIM (Molliq et al., 2009) requests a determination of Lagrange multiplier in the method, HAM does not need this request, Unlike ADM (Momani, 2005), where the calculation Adomian polynomials is needed to deal with nonlinear terms, We will apply the GA algorithm to get a better solution for the same equation. The choice of this technique to obtain approximate solution is motivated by the following factors. First, GA can provide solutions for highly complex search spaces also. Second, GA can be performed well approximating solutions to all types of problems. As a result, the approach does not make any assumptions for underlying the shape of the fitness function (Dass et al., 2014; Cheng et al., 2011; Li & Yin, 2014).

Remainder of the paper is orderly as follows: in section 2 we have some definitions of the fractional function. In section, 3 we have the basic ideas of the homotopy analysis method. In section 4, we have the genetic algorithm. In section 5, we have the proposed method (GA-HAM). In section 6, we have a test of convergence of homotopy analysis method. In section 7 we have examples (1 and 2) of the fractional heat–like equation. In section 8, we have an example (3) of the fractional wave–like Equations. Finally, section 9 concludes the paper.

2. Fractional Calculus

In this part, we give some definitions, lemmas and properties of the fractional calculus to enable us to follow the solutions of the problem given. These definitions include, Riemann-Liouville, Wey, Reize, Compos, Caputo, and Nashimoto fractional operators.

2.1 Definition

"Let $\alpha \in \mathbb{R}_+$. The operator J_a^α defined on the usual Lebesque space $L_1[a,b]$ by

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1} f(s)ds,$$
$$J_a^0 f(x) = f(x),$$

for $a \le x \le b$, is called the Riemann-Liouville fractional integral operator of order α.
Properties of the operator J^α can be found in (Luchko & Groreflo, 1998; Gorenflo & Mainardi, 1997), we mention the following:

For $f \in L_1[a,b], \alpha, \beta \ge 0$ and $\gamma > -1$,

1. J_a^α exists for almost every $x \in [a,b]$,
2. $J_a^\alpha J_a^\beta f(x) = J_a^{\alpha+\beta} f(x)$,
3. $J_a^\alpha J_a^\beta f(x) = J_a^\beta J_a^\alpha f(x)$,
4. $J_a^\alpha (x-a)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(x-a)^{\alpha+\gamma}$.

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall modify fractional differential operator D^α proposed by (Mainardi, 1997; Caputo, 1967).

2.2 Definition

"The fractional derivative of $f(x)$ in (Caputo, 1967) is defined as:

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)}\int_a^x (x-s)^{m-\alpha-1} f^{(m)}(s)ds,$$

for$m-1 < \alpha \le m,\ m \in \mathbb{N},\ x > 0$.
Also, we need here two of its basic properties."

2.3 Lemma

"If $m-1 < \alpha \le m$and $f \in L_1[a,b]$, then

$$D_a^\alpha J_a^\alpha f(x) = f(x),$$

and

$$J_a^\alpha D_a^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{(x-a)^k}{k!}, x > 0.$$

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem. In this paper, we consider the three-dimensional time fractional heat-like and wave-like Equation (1), where the unknown function $u = u(x,y,z,t)$ is a assumed to be a causal function of time, i.e., vanishing for $t < 0$, and the fractional derivative is taken in (Caputo, 1967).

2.4 Definition

"For m to be the smallest integer that exceeds α, the Caputo fractional derivative of order $\alpha > 0$ is defined as

$$D^{\alpha}f(x) = \frac{\partial^{\alpha}u(x,y,z,t)}{\partial t^{\alpha}}$$

$$= \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_{a}^{t}(t-s)^{m-\alpha-1}\frac{\partial^{m}u(x,y,z,t)}{\partial t^{m}}ds, & \text{for} \quad m-1 < \alpha \le m \\ \frac{\partial^{m}u(x,y,z,t)}{\partial t^{m}}, & \text{for} \quad \alpha = m \in \mathbb{N}. \end{cases}$$

3. Basic Ideas of the Homotopy Analysis Method

To illustrate this method, let us get the following equation

$$\mathcal{N}[y(x,t)] = k(x,t), \tag{4}$$

where $\mathcal{N}$ is an operator (nonlinear), x,t represent the independent variable, y(x,t) is a function (unknown) and k(x,t) is an analytic function. Through the generalizing, the classical homotopy method, Liao (1992, 1997, 1999) constructs the so-called zero-order deformation of the equation

$$(1-q)\mathcal{L}[\phi(x,t;q) - y_0(x,t)] = qh\{\mathcal{N}[\phi(x,t;q)] - k(x,t)\}, \tag{5}$$

Where$q \in [0,1]$ is an embed parameter, $\mathcal{L}$ is an adjuvant linear effect, h is a non-zero adjuvant function,$y_0(x,t)$ is an initial guess of $y(x,t)$,$\phi(x,t;q)$ is an unknown function. Note in this way that we have the freedom to choose adjuvant objects such as h and $\mathcal{L}$ in HAM (Al-Hayani, 2017) visibly, when $q = 0$ and $q = 1$, together

$$\phi(x,t;0) = y_0(x,t)\text{and}\phi(x,t;1) = y(x,t)$$

Suspension. Thus, as q increases from 0 to 1, the solution $\phi(x,t;q)$ varies from the initial guess $y_0(x,t)$ to the solution $y(x,t)$. Expanding $\phi(x,t;q)$ in Taylor series with respect to q, one has

$$\phi(x,t;q) = y_0(x,t) + \sum_{n=1}^{+\infty} y_n(x,t)q^n, \tag{6}$$

where

$$y_n = \frac{1}{n!}\frac{\partial^n\phi(x,t;q)}{\partial q^n}\bigg|_{q=0}. \tag{7}$$

If the adjuvant linear operator, the initial guess, the adjuvant parameter h, and the adjuvant function are so properly chosen, then the series (6) converges at $q = 1$ and one has

$$\phi(x,t;1) = y_0(x,t) + \sum_{n=1}^{+\infty} y_n(x,t),$$

whose ought be one of the solves of the authentic nonlinear equations proved by Liao (1992, 1997, 1999). If $h = -1$, equation (2) be

$$(1-q)\mathcal{L}[\phi(x,t;q)-y_0(x,t)]+q\{\mathcal{N}[\phi(x,t;q)]-k(x,t)\}=0. \quad (8)$$

According to (7), the governing equations can be deduced from the zeroth-order deformation equations (5). We define the vectors
$\vec{y_i}=\{y_0(x,t),y_1(x,t),\cdots,y_i(x,t)\}$.

Differentiating

$$(1-q)\mathcal{L}[\phi(x,t;q)-y_0(x,t)]=hqH(x,t)\mathcal{N}[\phi(x,t;q)], \quad (9)$$

$n$times with respect to the embedding parameter q and then setting $q=0$ and finally dividing them by $n!$, we have the so-called *nth*-order deformation equation

$$\mathcal{L}[y_n(x,t)-\chi_w y_{n-1}(x,t)]=hR_n\big(y_{n-1}(x,t)\big), \quad (10)$$

where

$$R_n\big(y_{n-1}(x,t)\big)=\frac{1}{(n-1)!}\frac{\partial^{n-1}\{\mathcal{N}[\phi(x,t;q)]-k(x,t)\}}{\partial q^{n-1}}\bigg|_{q=0}, \quad (11)$$

and

$$\chi_w=\begin{cases}0, & w\le 1,\\ 1, & w>1.\end{cases} \quad (12)$$

It should be emphasized that $y_n(x,t)$ $(n\ge 1)$ are governed by the linear equation (10) with the linear boundary conditions that come from the original problem, which can be easily solved by symbolic computation software such as Maple.

4. Genetic Algorithm (GA)

GA is an enhancement strategy, which is based on the natural selection and is an easy to implement, were presented in 1970 by John Holland. The space of all solutions is to find the search space. Each point in region of the search space contains one specific solution. Each solution can be represented by its fitness value (Zang et al., 2016; Cheng et al., 2011; Dass et al., 2014). GA is intended to imitate a natural activity; a significant part of the pertinent wording is coming from biology science (Al-Hayani, 2017).

Start: Initialize random population of chromosomes.
Fitness Functions: Evaluate the fitness (f) of each chromosome x in population.
New population: Create a new population and chromosomes by repeating the following steps.

- Reproduction (Selection): Select two parent chromosomes according to their fitness.
- Crossover: The operation is applied to the selected chromosomes. The chromosomes chosen by recombine with each other and new chromosomes will be created to form new offspring.
- Mutation: Some elements of the chromosomes introduce random changes in the genes to preserve the genetic diversity.

Replace: Use new produced population for a further keep running of the algorithm.
Termination: If the end condition is fulfilled, stop, and restore the best arrangement in current populace.

Loop: Go to step 2.

5. Proposed Method GA-HAM

The proposed method is based on finding the optimal parameters h in HAM (Heat–Like Equations one dimensional and two dimensional) and (wave–Like equations one dimensional) using the GA with the HAM. The result of HAM solution series is used to formulate the fitness function (F) in the GA using the following equations:

$$F(k) = \frac{1}{m}\sum_{j=1}^{m}\left(u(x_i, t_j) - \phi(x_i, t_j)\right)^2,$$

where m the total numbers of steps used in the domain of x and $t$respectively, ϕ the approximate solution of the Examples (1, 2 and 3), $u$are the exact solutions for the Examples (1, 2 and 3). Frepresents the fitness function (MSE) is solved by using the GA which involves three operators on its population (selection, crossover and mutation). The proposed method assumes a first standard model structure where the parameter is unknown. The goal of this strategy to locate the ideal parameter h(Heat–Like equations one dimensional and two dimensional) and (wave–Like equations one dimensional) which minimizes differences between the model out vector and the real response vector.

6. A Test of Convergence

The series

$$\boldsymbol{y_0(x,t) + \sum_{n=1}^{+\infty} y_n(x,t)},$$

is convergent, where $y_n(x,t)$ is governed by the high-order deformation equation (10), under the definitions (11) and (12), it must be a solution of equation (4).

7. Fractional Heat–Like Equations

In this section, we explain our analysis by testing for the following two fractional heat-like equation.

7.1 Example 1

Firstly, we consider the one-dimensional initial boundary value problems

$$D_t^{\alpha} = \frac{1}{2}x^2 u_{xx}, \quad 0 < x < 1, \quad 0 < \alpha \leq 1, \quad t > 0\,, \tag{13}$$

subject to the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = e^t \tag{14}$$

and the initial condition

$$u(x,0) = x^2.$$

The exact solution for this problem is

$$u_{Exact}(x,t) = x^2 e^t. \tag{15}$$

To solve (13) and (14) by the standard HAM, we select the initial approximation
$u(x,0) = x^2$
For $0 < \alpha \leq 1$, and utilizingthe initial condition we put

$$u_0(x,t) = \sum_{k=0}^{0} \frac{\partial^k}{\partial t^k} u(x,0^+) \frac{t^k}{k!} = x^2.$$

Although, we have freedom to choose an initial guess, one has

$$J_x^\alpha(D_t^\alpha) = J_x^\alpha\left(\frac{1}{2}x^2 u_{xx}\right). \tag{16}$$

According to the Eqs. (15) and

$$J_x^\alpha D_t^\alpha[u_n(x,t) - \chi_w u_{n-1}(x,t)] = hJ_x^\alpha\{R_n[\vec{u}_{n-1}(x,t)]\}, \ \ w \geq 1 \tag{17}$$

where

$$R_n[\vec{u}_{n-1}(x,t)] = (D_t^\alpha) - \left(\frac{1}{2}x^2 u_{xx}\right).$$

We will get iterations

$$u_1(x,t) = -\frac{hx^2t^\alpha}{\Gamma(\alpha+1)}$$

$$u_2(x,t) = \left(-\frac{ht^\alpha}{\Gamma(\alpha+1)} - \frac{h^2t^\alpha}{\Gamma(\alpha+1)} + \frac{h^2t^{2\alpha}\sqrt{\pi}}{(2)^{2\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)}\right)x^2$$

$$u_3(x,t) = \left(-\frac{ht^\alpha}{\Gamma(\alpha+1)} - \frac{2h^2t^\alpha}{\Gamma(\alpha+1)} + \frac{2h^2t^{2\alpha}\sqrt{\pi}}{(2)^{2\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)} - \frac{h^3t^\alpha}{\Gamma(\alpha+1)} + \frac{2h^3t^{2\alpha}\sqrt{\pi}}{(2)^{2\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)} - \frac{2}{3}\frac{h^3t^{3\alpha}\pi\sqrt{3}}{(3)^{3\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{3}\right)\Gamma\left(\alpha+\frac{2}{3}\right)}\right)x^2$$

$$\vdots$$

So, the solution approximate in a series form is

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{+\infty} u_n(x,t),$$

we get

$$\phi(x,t) = \left(1 - \frac{4ht^\alpha}{\Gamma(\alpha+1)} - \frac{6h^2t^\alpha}{\Gamma(\alpha+1)} + \frac{6h^2t^{2\alpha}\sqrt{\pi}}{(2)^{2\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)} - \frac{4h^3t^\alpha}{\Gamma(\alpha+1)} + \frac{8h^3t^{2\alpha}\sqrt{\pi}}{(2)^{2\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)} - \frac{4}{3}\frac{h^3t^{3\alpha}\pi\sqrt{3}}{(3)^{3\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{3}\right)\Gamma\left(\alpha+\frac{2}{3}\right)} - \frac{h^4t^\alpha}{\Gamma(\alpha+1)} + \frac{3h^4t^{2\alpha}\sqrt{\pi}}{(2)^{2\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)} - \frac{2\pi\sqrt{3}h^4t^{3\alpha}}{3^{3\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{3}\right)\Gamma\left(\alpha+\frac{2}{3}\right)} - \frac{4}{3}\frac{h^3t^{3\alpha}\pi\sqrt{3}}{(3)^{3\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{3}\right)\Gamma\left(\alpha+\frac{2}{3}\right)} + \frac{\pi^{\left(\frac{3}{2}\right)}\sqrt{2}h^4t^{4\alpha}}{(4)^{4\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\alpha+\frac{1}{4}\right)\Gamma\left(\alpha+\frac{3}{4}\right)} + \cdots\right)x^2.$$

When $n \to \infty$ and $\alpha = 1$, the above series has the closed form
$u_{Exact}(x,t) = x^2e^t$.

In Tables 1-6, we list the optimal value of h by GA, value of h in Classical Method and the MSE for Example 1within the interval $0 \leq x \leq 1$ and $0 \leq t \leq 1$.

Table 1: When $\alpha = 1/4$ for Example 1

Optimal h by GA	MSE
-0.2531	$2.6238e - 02$

Table 2: When $\alpha = 1/4$ for Example 1

h in Classical Method	MSE
-0.1	$8.4616e - 02$
-0.2	$3.3034e - 02$
-0.3	$3.1320e - 02$
-0.4	$7.4405e - 02$
-0.5	$1.5732e - 01$
-0.6	$2.7518e - 01$
-0.7	$4.2322e - 01$
-0.8	$5.9680e - 01$
-0.9	$7.9140e - 01$
-1.0	$1.0026e - 00$

Table 3: When $\alpha = 1/2$ for Example 1

Optimal h by GA	MSE
-0.33184	$1.0135e - 02$

Table 4: When $\alpha = 1/2$ for Example 1

h in classical method	MSE
-0.1	$8.9154e - 02$
-0.2	$3.2199e - 02$
-0.3	$1.0850e - 02$
-0.4	$1.6714e - 02$
-0.5	$4.2443e - 02$
-0.6	$8.1679e - 02$
-0.7	$1.2900e - 01$
-0.8	$1.7986e - 01$
-0.9	$2.3057e - 01$
-1.0	$2.7822e - 01$

Table 5: When $\alpha = 3/4$ for Example 1

Optimal h by GA	MSE
-0.48504	$2.8874e - 03$

Table 6: When $\alpha = 3/4$ for Example 1

h in classical method	MSE
-0.1	$9.9205e-02$
-0.2	$4.4144e-02$
-0.3	$1.5155e-02$
-0.4	$3.7206e-03$
-0.5	$3.3227e-03$
-0.6	$9.1100e-03$
-0.7	$1.7610e-02$
-0.8	$2.6473e-02$
-0.9	$3.4245e-02$
-1.0	$4.0172e-02$

In the previous Tables 1-6, we note that the value of h which was obtained from the GA in Example 1 gives better results for the MSE than all the values of h that were selected using the classical method, although several values were chosen for α. In Figures (1, 2 and 3), we represent the exact solution with a continuous line, the symbol $+$ for the HAM and the symbol o for the proposed method within the interval $0 \le x \le 1$.

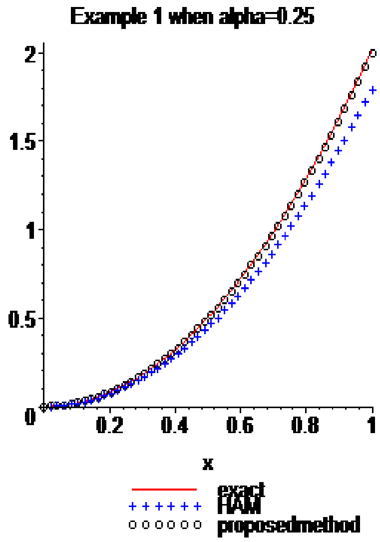

Figure 1: When $\alpha = 0.25$ for Example 1

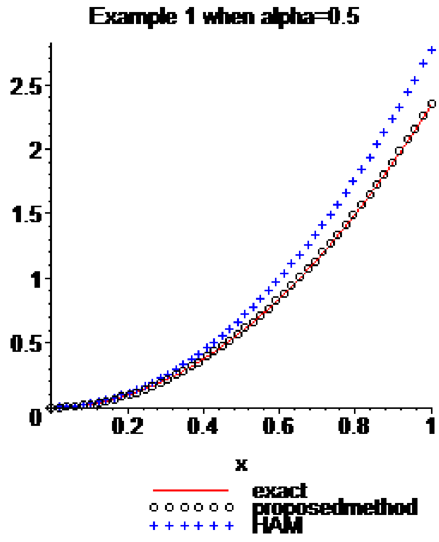

Figure 2: When $\alpha = 0.5$ for Example 1

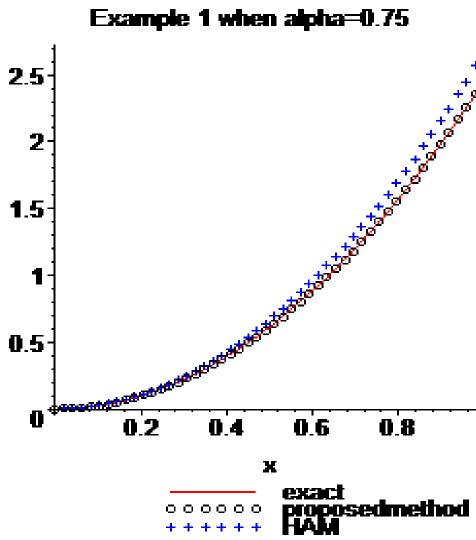

Figure 3: When $\alpha = 0.75$ for Example 1

7.2 Example 2

We consider the two- dimensional initial boundary value problems

$$D_t^\alpha = \frac{1}{2}\left(y^2 u_{xx} + x^2 u_{yy}\right), \quad 0 < x, y < 1, \quad 0 < \alpha \leq 1, \quad t > 0 \ , \tag{18}$$

Subject to the Neumann boundary conditions

$$u_x(0, y, t) = 0, \quad u_x(1, y, t) = 2sinht, \tag{19}$$
$$u_y(x, 0, t) = 0, \quad u_y(x, 1, t) = 2cosht,$$

and the initial condition

$$u(x, y, 0) = y^2.$$

The exact solution for this problem is

$$u(x, y, t) = x^2 sinht + y^2 cosht \tag{20}$$

To solve (18) and (19) by the standard HAM, we select the initial approximation

$u(x, y, 0) = y^2$

For $0 < \alpha \leq 1$,and utilizing the initial condition we put

$$u_0(x, y, t) = \sum_{k=0}^{0} \frac{\partial^k}{\partial t^k} u(x, y, 0^+) \frac{t^k}{k!} = y^2,$$

Although, we have freedom to choose an initial guess, one has

$$J_t^\alpha(D_t^\alpha) - J_t^\alpha\left(\frac{1}{2}\left(y^2 u_{xx} + x^2 u_{yy}\right)\right), \tag{21}$$

According to the Eqs. (20) and

$$J_t^{\alpha} D_t^{\alpha}[u_n(x,y,t) - \chi_w u_{n-1}(x,y,t)] = hJ_t^{\alpha}\{R_n[\vec{u}_{n-1}(x,y,t)]\}, \ \ w \geq 1 \tag{22}$$

where

$R_n[\vec{u}_{n-1}(x,y,t)] = (D_t^{\alpha}) - \left(\frac{1}{2}\left(y^2 u_{xx} + x^2 u_{yy}\right)\right).$

We will get iterations

$$u_1(x,y,t) = -\frac{hx^2t^{\alpha}}{\Gamma(\alpha+1)}$$

$$u_2(x,y,t) = -\frac{hx^2t^{\alpha}}{\Gamma(\alpha+1)} - \frac{h^2x^2t^{\alpha}}{\Gamma(\alpha+1)} + \frac{h^2y^2t^{2\alpha}\sqrt{\pi}}{(2)^{2\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)}$$

$$u_3(x,y,t) = -\frac{hx^2t^{\alpha}}{\Gamma(\alpha+1)} - \frac{2h^2x^2t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2h^2y^2t^{2\alpha}\sqrt{\pi}}{(2)^{2\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)} - \frac{h^3x^2t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2h^3y^2t^{2\alpha}\sqrt{\pi}}{(2)^{2\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)} - \frac{2}{3}\frac{h^3x^2t^{3\alpha}\pi\sqrt{3}}{(3)^{3\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{3}\right)\Gamma\left(\alpha+\frac{2}{3}\right)}$$

$$\vdots$$

So, the solve approximate in a chain form is

$$u(x,y,t) = u_0(x,y,t) + \sum_{n=1}^{+\infty} u_n(x,y,t),$$

we get

$$\phi(x,y,t) = y^2 - \frac{h^4x^2t^{\alpha}}{\Gamma(\alpha+1)} - \frac{4h^3x^2t^{\alpha}}{\Gamma(\alpha+1)} + \frac{3h^4y^2t^{2\alpha}\sqrt{\pi}}{(2)^{2\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)} - \frac{6h^2x^2t^{\alpha}}{\Gamma(\alpha+1)} - \frac{2\pi h^4x^2t^{3\alpha}\sqrt{3}}{(3)^{3\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{3}\right)\Gamma\left(\alpha+\frac{2}{3}\right)} + \frac{8h^3y^2t^{2\alpha}\sqrt{\pi}}{(2)^{2\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)} - \frac{4hx^2t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\pi^{\left(\frac{3}{2}\right)}\sqrt{2}h^4y^2t^{4\alpha}}{(4)^{4\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\alpha+\frac{1}{4}\right)\Gamma\left(\alpha+\frac{3}{4}\right)} - \frac{8}{3}\frac{\pi\sqrt{3}h^3x^2t^{3\alpha}}{3^{3\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{3}\right)\Gamma\left(\alpha+\frac{2}{3}\right)} + \frac{6\sqrt{\pi}h^2y^2t^{2\alpha}}{(2)^{2\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)} + \cdots$$

When $n \to \infty$ and $\alpha = 1$, the above series has the closed form
$u_{Exact}(x,y,t) = x^2 sinht + y^2 cosht.$

In Tables 7-12, we list the optimal value of *h* by GA, value of *h* in Classical Method and the MSE for Example 2.

Table 7: When$\alpha = 1/4$, and $x = 1/2$ for Example 2

Optimal *h* by GA	MSE
-0.41511	$1.6110\ e-01$

Table 8: When $\alpha = 1/4$, and $x = 1/2$ for Example 2

h in Classical Method	MSE
-0.1	$2.2001e-01$
-0.2	$1.8654e-01$
-0.3	$1.6733e-01$
-0.4	$1.6101e-01$
-0.5	$1.6626e-01$
-0.6	$1.8175e-01$
-0.7	$2.0622e-01$
-0.8	$2.3841e-01$
-0.9	$2.7709e-01$
-1.0	$3.2109e-01$

Table 9: When $\alpha = 1/2$, and $x = 1/2$ for Example 2

Optimal h by GA	MSE
-0.52268	$1.5636e-01$

Table 10: When $\alpha = 1/2$, and $x = 1/2$ for Example 2

h in classical method	MSE
-0.1	$2.2395e-01$
-0.2	$1.9260e-01$
-0.3	$1.7243e-01$
-0.4	$1.6112e-01$
-0.5	$1.5662e-01$
-0.6	$1.5716e-01$
-0.7	$1.6121e-01$
-0.8	$1.6746e-01$
-0.9	$1.7486e-01$
-1.0	$1.8253e-01$

Table 11: When $\alpha = 3/4$, and $x = 1/2$ for Example 2

Optimal h by GA	MSE
-1.614	$1.5674e-01$

Table 12: When$\alpha = 3/4$, and $x = 1/2$ for Example 2

h in classical method	MSE
-0.1	$2.2961e-01$
-0.2	$2.0222e-01$
-0.3	$1.8380e-01$
-0.4	$1.7189e-01$
-0.5	$1.6455e-01$
-0.6	$1.6033e-01$
-0.7	$1.5810e-01$
-0.8	$1.5710e-01$
-0.9	$1.5676e-01$
-1.0	$1.5675e-01$

In the previous Tables 7-12, we note that the value of h which was obtained from the GA in Example 2 gives better results for the MSE than all the values of h that were selected using the classical method, although several values were chosen for α. In Figures (4, 5 and 6), we represent the exact solution with a continuous line, the symbol $+$ for the HAM and the symbol o for the proposed method within the interval $0 \leq x \leq 1$.

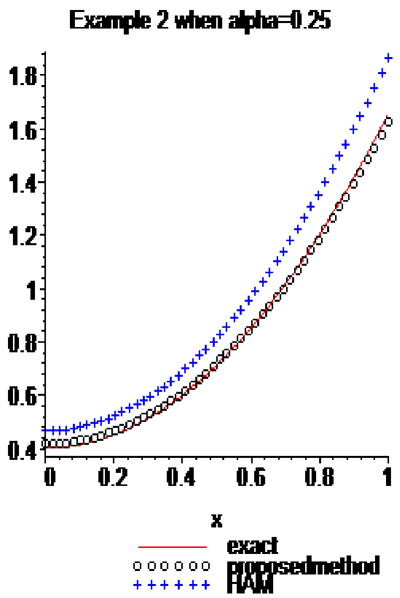

Figure 4: When $\alpha = 0.25$ for Example 2

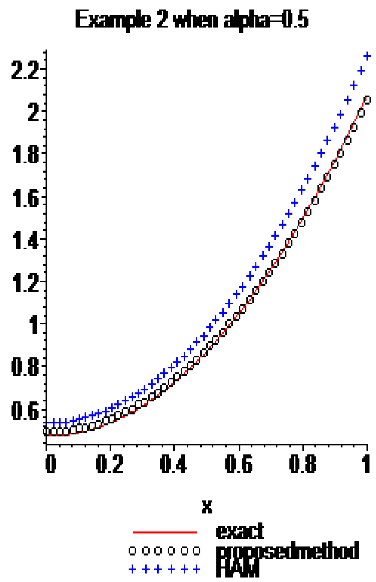

Figure5: When $\alpha = 0.5$ for Example 2

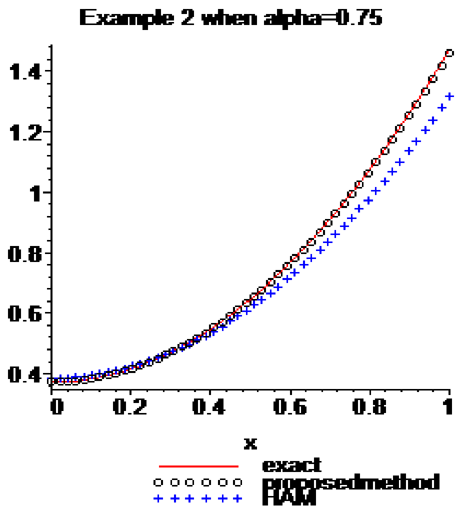

Figure 6: When $\alpha = 0.75$ for Example 2

8. Fractional Wave–Like Equations

In this section, we demonstrate our analysis by the survey for following one fractional wave-like equation.

8.1 Example 3

Finally, we consider the one-dimensional initial boundary value problems

$$D_t^\alpha = \frac{1}{2}x^2 u_{xx}, \quad 0 < x < 1, \quad 1 < \alpha \le 2, \quad t > 0\,, \tag{23}$$

subject to the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 1 + sinht, \tag{24}$$

and the initial condition

$$u(x,0) = x, \quad u_t(x,0) = x^2.$$

The exact solution for this problem is

$$u_{Exact}(x,t) = x + x^2 sinht \tag{25}$$

To solve (23) and (24) by the standard HAM, we select the initial approximation

$$u(x,0) = x + x^2 t$$

For$1 < \alpha \le 2$,and utilizing the initial condition we put

$$u_0(x,t) = \sum_{k=0}^{1} \frac{\partial^k}{\partial t^k} u(x,0^+)\frac{t^k}{k!} = x + x^2 t,$$

Although, we have freedom to choose an initial guess, one has

$$J_x^\alpha (D_t^\alpha) = J_x^\alpha \left(\frac{1}{2}x^2 u_{xx}\right), \tag{26}$$

According to the Eqs. (25) and

$$J_x^\alpha D_t^\alpha[u_n(x,t) - \chi_w u_{n-1}(x,t)] = hJ_x^\alpha\{R_n[\vec{u}_{n-1}\,(x,t)]\},\;\; w \geq 1 \tag{27}$$

where

$R_n[\vec{u}_{n-1}\,(x,t)] = (D_t^\alpha) - \left(\frac{1}{2}x^2 u_{xx}\right),$

We will get iterations

$$u_1(x,t) = -\frac{hx^2t^{\alpha+1}}{\Gamma(\alpha+1)+\Gamma(\alpha)\alpha^2}$$

$$u_2(x,t) = \left(-\frac{ht^{\alpha+1}}{\Gamma(\alpha+1)+\Gamma(\alpha)\alpha^2} - \frac{h^2t^{\alpha+1}}{\Gamma(\alpha+1)+\Gamma(\alpha)\alpha^2} + \frac{h^2t^{2\alpha+1}\Gamma(\alpha+1)}{(\Gamma(\alpha+1)+\Gamma(\alpha)\alpha^2)\left(\frac{2(2^{2\alpha})\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)\alpha^2}{\sqrt{\pi}} + \frac{(2^{2\alpha})\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)\alpha}{\sqrt{\pi}}\right)} + \frac{h^2t^{2\alpha+1}\Gamma(\alpha)\alpha^2}{(\Gamma(\alpha+1)+\Gamma(\alpha)\alpha^2)\left(\frac{2(2^{2\alpha})\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)\alpha^2}{\sqrt{\pi}} + \frac{(2^{2\alpha})\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)\alpha}{\sqrt{\pi}}\right)}\right)x^2$$

$$u_3(x,t) = \left(-\frac{ht^{\alpha+1}}{\Gamma(\alpha+1)+\Gamma(\alpha)\alpha^2} - \frac{h^2t^{\alpha+1}}{\Gamma(\alpha+1)+\Gamma(\alpha)\alpha^2} \right.$$

$$+ \frac{2h^2t^{2\alpha+1}\Gamma(\alpha+1)}{(\Gamma(\alpha+1)+\Gamma(\alpha)\alpha^2)\left(\frac{2(2^{2\alpha})\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)\alpha^2}{\sqrt{\pi}} + \frac{(2^{2\alpha})\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)\alpha}{\sqrt{\pi}}\right)}$$

$$+ \frac{2h^2t^{2\alpha+1}\Gamma(\alpha)\alpha^2}{(\Gamma(\alpha+1)+\Gamma(\alpha)\alpha^2)\left(\frac{2(2^{2\alpha})\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)\alpha^2}{\sqrt{\pi}} + \frac{(2^{2\alpha})\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)\alpha}{\sqrt{\pi}}\right)} - \frac{h^3t^{\alpha+1}}{\Gamma(\alpha+1)+\Gamma(\alpha)\alpha^2}$$

$$+ \frac{2h^3t^{2\alpha+1}\Gamma(\alpha+1)}{(\Gamma(\alpha+1)+\Gamma(\alpha)\alpha^2)\left(\frac{2(2^{2\alpha})\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)\alpha^2}{\sqrt{\pi}} + \frac{(2^{2\alpha})\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)\alpha}{\sqrt{\pi}}\right)}$$

$$+ \frac{2h^3t^{2\alpha+1}\Gamma(\alpha)\alpha^2}{(\Gamma(\alpha+1)+\Gamma(\alpha)\alpha^2)\left(\frac{2(2^{2\alpha})\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)\alpha^2}{\sqrt{\pi}} + \frac{(2^{2\alpha})\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)\alpha}{\sqrt{\pi}}\right)}$$

$$- \frac{3h^3t^{3\alpha+1}(2^{2\alpha})\Gamma(\alpha)^2\Gamma\left(\alpha+\frac{1}{2}\right)\alpha^3}{\left(\frac{3}{2}\frac{(3^{3\alpha})\sqrt{3}\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{3}\right)\Gamma\left(\alpha+\frac{2}{3}\right)\alpha^2}{\pi} + \frac{1}{2}\frac{(3^{3\alpha})\sqrt{3}\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{3}\right)\Gamma\left(\alpha+\frac{2}{3}\right)\alpha}{\pi}\right)\sqrt{\pi}\Gamma(\alpha+1)+\Gamma(\alpha)\alpha^2}$$

$$\left. + \cdots \right) x^2$$

$$\vdots$$

Thus, the solution approximate in a series form is given by

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{+\infty} u_n(x,t),$$

we

ge $$\phi(x,t) = \left(t - \frac{4ht^{\alpha+1}}{\Gamma(\alpha+1)+\Gamma(\alpha)\alpha^2} - \frac{6h^2t^{\alpha+1}}{\Gamma(\alpha+1)+\Gamma(\alpha)\alpha^2} - \frac{4h^3t^{\alpha+1}}{\Gamma(\alpha+1)+\Gamma(\alpha)\alpha^2} - \right.$$

$$\frac{h^4t^{\alpha+1}}{\Gamma(\alpha+1)+\Gamma(\alpha)\alpha^2} + \frac{6h^2t^{2\alpha+1}\Gamma(\alpha+1)}{(\Gamma(\alpha+1)+\Gamma(\alpha)\alpha^2)\left(\frac{2(2^{2\alpha})\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)\alpha^2}{\sqrt{\pi}} + \frac{(2^{2\alpha})\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)\alpha}{\sqrt{\pi}}\right)} +$$

$$\frac{6h^2t^{2\alpha+1}\Gamma(\alpha)\alpha^2}{(\Gamma(\alpha+1)+\Gamma(\alpha)\alpha^2)\left(\frac{2(2^{2\alpha})\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)\alpha^2}{\sqrt{\pi}}+\frac{(2^{2\alpha})\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)\alpha}{\sqrt{\pi}}\right)}+$$

$$\frac{8h^2t^{2\alpha+1}\Gamma(\alpha+1)}{(\Gamma(\alpha+1)+\Gamma(\alpha)\alpha^2)\left(\frac{2(2^{2\alpha})\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)\alpha^2}{\sqrt{\pi}}+\frac{(2^{2\alpha})\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)\alpha}{\sqrt{\pi}}\right)}+$$

$$\left.\frac{8h^2t^{2\alpha+1}\Gamma(\alpha)\alpha^2}{(\Gamma(\alpha+1)+\Gamma(\alpha)\alpha^2)\left(\frac{2(2^{2\alpha})\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)\alpha^2}{\sqrt{\pi}}+\frac{(2^{2\alpha})\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)\alpha}{\sqrt{\pi}}\right)}+\cdots\right)x^2$$

When $n \to \infty$ and $\alpha = 2$, the above series has the closed form

$$u_{Exact}(x,t) = x + x^2 sinht.$$

In Tables 13-18, we list the optimal value of *h* by GA, value of *h* in Classical Method and the MSE for Example 3 within the interval $0 \le x \le 1$ and $0 \le t \le 1$.

Table 13: When $\alpha = 1.25$ for Example 3

Optimal *h* by GA	MSE
-0.12822	$2.7370\ e-05$

Table 14: When $\alpha = 1.25$ for Example 3

h in Classical Method	MSE
-0.1	$3.7062\ e-05$
-0.2	$3.9027\ e-04$
-0.3	$1.3808\ e-03$
-0.4	$2.4823\ e-03$
-0.5	$3.4344\ e-03$
-0.6	$4.1420\ e-03$
-0.7	$4.6035\ e-03$
-0.8	$4.8635\ e-03$
-0.9	$4.9822\ e-03$
-1.0	$5.0184\ e-03$

Table 15: When $\alpha = 1.5$ for Example 3

Optimal *h* by GA	MSE
-0.19438	$1.9794e-05$

Table 16: When $\alpha = 1.5$ for Example 3

h in classical method	MSE
-0.1	$1.6752\ e-04$
-0.2	$2.6153\ e-05$
-0.3	$2.8209\ e-04$
-0.4	$6.3827\ e-04$
-0.5	$9.5438\ e-04$
-0.6	$1.1818\ e-03$
-0.7	$1.3202\ e-03$
-0.8	$1.3900\ e-03$
-0.9	$1.4166\ e-03$
-1.0	$1.4221\ e-03$

Table 17: When $\alpha = 1.75$ for Example 3

Optimal h by GA	MSE
-0.29684	$4.1098e-06$

Table 18: When $\alpha = 1.75$ for Example 3

h in classical method	MSE
-0.1	$3.6096\ e-04$
-0.2	$4.9148\ e-05$
-0.3	$4.8270\ e-06$
-0.4	$5.8645\ e-05$
-0.5	$1.2888\ e-04$
-0.6	$1.8369\ e-04$
-0.7	$2.1669\ e-04$
-0.8	$2.3222\ e-04$
-0.9	$2.3735\ e-04$
-1.0	$2.3814\ e-04$

In the previous tables 13-18, we note that the value of h which was obtained from the GA in Example 3 gives better results for the MSE than all the values of h that were selected using the classical method, although several values were chosen for α. In Figures (7, 8 and 9), we represent the exact solution with a continuous line, the symbol $+$ for the HAM and the symbol o for the proposed method within the interval $0 \leq x \leq 1$.

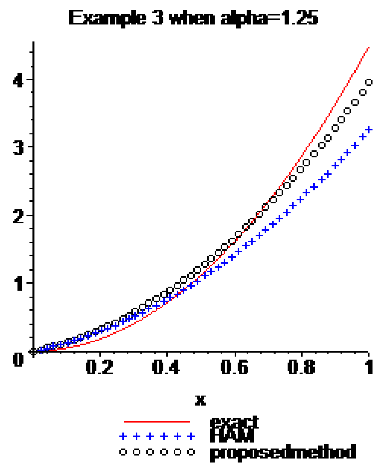

Figure 7: When $\alpha = 1.25$ for Example 3

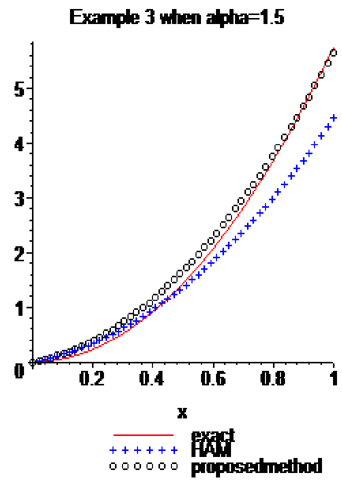

Figure 8: When $\alpha = 1.5$ for Example 3

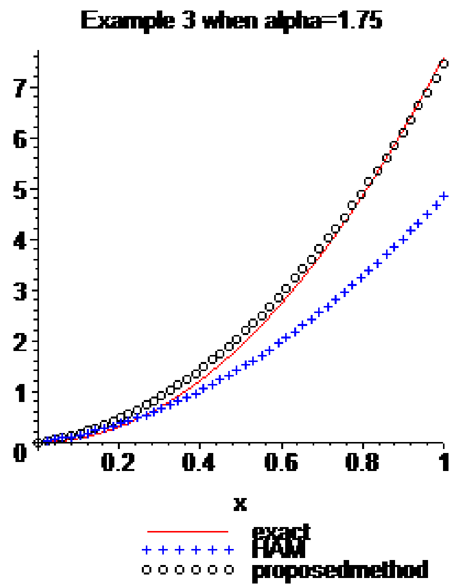

Figure 9: When $\alpha = 1.75$ for Example 3

8. Conclusion

From the results of this research, we can say that the HAM is successful in giving results close to the exact solution of the equations of fractional heat-like and wave-like. The basic idea of GA in this proposed method is to find the optimal parameters of the fractional heat-like and wave-like equations as it can be widely applied to nonlinear systems and to solve for any kind of equations.

References

Agrawal, O. (2002). Solution for a fractional diffusion-wave equation defined in a bounded domain. *Nonlinear Dynam*, 29, 145-155.

Al-Hayani, W. (2017). Daftardar-Jafari method for fractional heat-like and wave-like equations with variable coefficients. *Applied Mathematics*, 8, 215-228.

Caputo, M. (1967). Linear models of dissipation whose Q is almost frequency independent. Part II. *J.*

Roy. Astral. Soc. 13, 529-539.

Cheng,W., Shi, H., Xin, X., Li, D. (2011). An elitism strategy based genetic algorithm for streaming pattern discovery in wireless sensor networks. *Commun. Lett. IEEE*, 15, 419–421.

Dass, M. V., Ali, M. R., & Ali, M. M. (2014). Image Retrieval Using Interactive Genetic Algorithm, IEEE, International Conference on Computational Science and Computational Intelligence.

Duld, A. (1997). *Lecture on algebraic topology*. Berlin: Springer, Berlin.

Fujita, Y. (1990). Cauchy problems of fractional order and stable processes. *Japan J. Appl. Math*. 7 (3), 459-476.

Gorenflo, R., & Mainardi, F. (1997). Fractional calculus: Integral and differential equations of fractional order. In A. Carpinteri, F. Mainardi (Eds.), *Fractals and fractional calculus*. New York.

Hilfer, R. (1995). Foundations of fractional dynamics. *Fractals,* 3 (3), 549-556.

Hilfer, R. (2000). Fractional diffusion based on Riemann-Liouville fractional derivative. *J. Phys. Chem*. 104, 3914-3917.

Kilbas, A., Srivastava, H., & Trujillo, J. (2006). *Theory and applications of fractional differential equations.* Elsevier, Holland.

Klafter, J., Blumen, A., & Shlesinger, M. (1984). Fractal behavior in trapping and reaction: A random walk study. *J. Stat. Phys*, 36, 561-578.

Li, X., & Yin, M. (2014). Parameter estimation for chaotic systems by hybrid differential evolution algorithm and artificial bee colony algorithm. *Nonlinear Dynamics*, 77(1-2) 61-71.

Liao, S. J. (1992). The proposed homotopy analysis technique for the solution of nonlinear problems, Doctoral dissertation, Shanghai Jiao Tong University.

Liao, S. J. (1997). An approximate solution technique not depending on small parameters Part 2: An application in fluid mechanics. *Int. J. Nonlinear Mech*. 32(5), 815-822.

Liao, S. J. (1999). An explicit, totally analytic approximation of Blasius' viscous flow problems. *International Journal of Nonlinear Mechanics*, 34 (4), 759-778.

Luchko, A.Y., & Groreflo, Y. (1998). The initial value problem for some fractional differential equations with the Caputo derivative, Preprint series A08-98, fachbreichmathematik und informatik, FreicUniversitat Berlin.

Mainardi, F. (1997). Fractional calculus: Some basic problems in continuum and statistical mechanics. In A. Carpinteri, F. Mainardi (Eds.), *Fractal and fractional calculus in continuum mechanics.* Springer-Verlag, New York.

Metzler, R., & Klafter, J. (2000). Boundary value problems fractional diffusion equations. *Physica A*, 278, 107-125.

Miller, K., & Ross, B. (1993). *An introduction to the fractional calculus and fractional differential equations*. New York: John Wiley and Sons, Inc.

Molliq, Y., Noorani, M., & Hashim, I. (2009). Variational iteration method for fractional heat- and wave-like equations. *Nonlinear Anal.: Real World Appl*. 10, 1854-1869.

Momani, S. (2005). Analytical approximate solution for fractional heat-like and wave-like equations with variable coefficients using the decomposition method. *Appl. Math. Comput*. 165, 459-472.

Oldham, K., & Spanier, J. (1974). *The fractional calculus*. New York: Academic Press.

Podlubny, I. (1999). *Fractional differential equations*. New York: Academic Press.

Zang,W. K., Sun, M. H., Jiang, Z. N. (2016). A DNA genetic algorithm inspired by biological membrane structure. *J. Comput. Theor. Nanosci*, 13, 3763–3772.