Construction of Bivariate F-Control Chart with Application

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Abstract: In this paper, construction of bivariate F-Control Chart through three stages (test of bivariate normal distribution, construction of bivariate S-Chart and use the relationship between the distribution of T^2 and F) corresponding for Shewart T^2-Chart for quality control on measurable properties has been suggested. F-Chart is characterized by minimizing general and total variance for the least possible to obtain the best quality control limits. And the installation of the control chart on the data representing the quality properties of yield stress and elongation for steel product from factory (Erbil Steel) depending on the program language MATLAB has been designed and statistical program SPSS has been employed. The paper found the efficiency of the proposed chart and the possibility of using it to control future data (phase-2).

Keywords: Bivariate Average Chart, Quality Control Chart, Hotelling's T^2

1. Introduction

Quality Control Charts have historically been used to monitor product quality in a production or manufacturing environment. Their general purpose is to provide information that can be used to uncover discrepancies or systematic patterns by comparing expected variance and observed variance. In a production environment it is important to improve product quality and productivity in order to maximize a company's profits (Deming, 1982).

There are many different variations of control charts that can be used to detect when processes go out of control. The most common and easily interpretable of these is the Shewart control chart. These charts, named after Walter Shewart, were created from an assumption that every process that has variation can be understood and statistically monitored (Savić, 2006). A Shewart chart includes three horizontal lines, a center line, an upper limit, and a lower limit and is the basis for all control charts. The center line serves as a baseline and is typically the expected value or the mean value, while the upper and lower limits are depicted by baselines and are evenly spaced below and above the baseline.

From another side, it is a fact of life that most data are naturally multivariate (or Bivariate). Hotelling in 1947 introduced a statistic which uniquely lends itself to plotting multivariate observations (Kovach, 2007). This statistic, appropriately named Hotelling's T^2, is a scalar that combines information from the dispersion and mean of several variables. Due to the fact that computations are laborious and fairly complex and require some knowledge of matrix algebra, acceptance of multivariate control charts by industry was slow and hesitant,

In this research, consideration will be given to the bivariate normal distribution and homogeneity of the data when the proposed panel is created, add to use the relationship between the distribution of T^2 and F.

2. Methodology

2.1 The Hotelling T^2 Control Chart

We have two quality characteristics x_1 and x_2 are jointly distributed according to the bivariate normal distribution (x_1 and x_2 are correlated). Let μ_1 and μ_2 be the mean values of the quality characteristics, and let σ_1^2 and σ_2^2 be the variance of x_1 and x_2, respectively. The covariance between x_1 and x_2 is denoted by σ_{12}. Therefore, the bivariate normal probability density function is (Morrison, 1976).

$$f(\underline{x}) = \frac{1}{(2\pi)|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\underline{x}-\underline{\mu})' \Sigma^{-1} (\underline{x}-\underline{\mu})\right\} \quad \cdots \quad (1)$$

$$-\infty \le \underline{x} \le \infty \quad , -\infty \le \underline{\mu} \le \infty \qquad and \quad \Sigma \ge 0$$

Where $\underline{x}' = (x_1 \quad x_2)$, $\underline{\mu}' = (\mu_1 \quad \mu_2)$ and $\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}$

The bivariate normal probability density function is as in the following figure (Douglas, 2009).

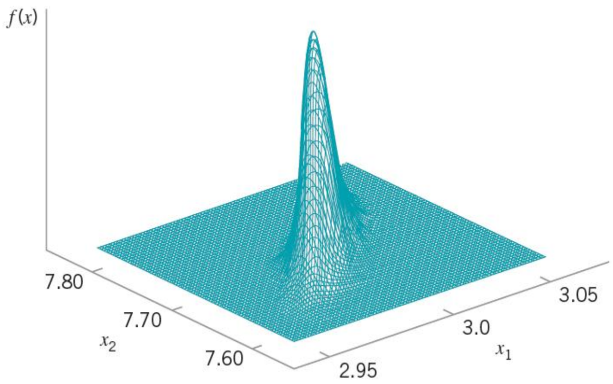

Figure 1: Bivariate normal probability density function

We assume that σ_1^2 , σ_2^2 and σ_{12} are unknown and that *m* such samples are available. The sample means, variances and covariance are calculated from each sample as usual; that is (Douglas, 2009).

$$\bar{x}_{jk} = \frac{1}{n}\sum_{i=1}^{n} x_{ijk} \qquad \begin{Bmatrix} j = 1,2 \\ k = 1,2,\cdots,m \end{Bmatrix} \qquad \cdots \quad (2)$$

$$S_{jk}^{2} = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_{ijk} - \bar{x}_{jk}\right)^{2} \qquad \cdots \quad (3)$$

Where x_{ijk} is the *i*th observation on the *j*th quality characteristic in the *k*th sample (Kuvattana, 2016). The covariance between quality characteristic *j* and quality characteristic *h* in the *k*th sample is:

$$S_{jhk} = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_{ijk} - \bar{x}_{jk}\right)\left(x_{ihk} - \bar{x}_{hk}\right) \qquad \{j \neq h\} \qquad \cdots \quad (4)$$

For average over all m samples, we have:

$$\bar{\bar{x}}_{j} = \frac{1}{m}\sum_{k=1}^{m} \bar{x}_{jk} \qquad \cdots \quad (5)$$

$$\bar{S}_{j}^{2} = \frac{1}{m}\sum_{k=1}^{m} S_{jk}^{2} \qquad \cdots \quad (6)$$

$$\bar{S}_{jh} = \frac{1}{m}\sum_{k=1}^{m} S_{jhk} \qquad \cdots \quad (7)$$

Where: $\underline{\bar{x}}'_{k} = \left(\bar{x}_{1k} \quad \bar{x}_{2k}\right)$, $\underline{\bar{\bar{x}}}' = \left(\bar{\bar{x}}_{1} \quad \bar{\bar{x}}_{2}\right)$ and $S = \begin{bmatrix} \bar{S}_{1}^{2} & \bar{S}_{12} \\ \bar{S}_{21} & \bar{S}_{2}^{2} \end{bmatrix}$

The test statistic for each sample means (Univariate) is:

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}}$$

By squaring the two sides we get:

$$t^{2} = \frac{\left(\bar{x} - \mu\right)^{2}}{S^{2}/n} = n\left(\bar{x} - \mu\right)\left(S^{2}\right)^{-1}\left(\bar{x} - \mu\right)$$

By using the matrices and for $\left(k = 1,2,\cdots,m\right)$ we get:

$$T_{k}^{2} = n\left(\underline{\bar{x}}_{k} - \underline{\bar{\bar{x}}}\right)' S^{-1}\left(\underline{\bar{x}}_{k} - \underline{\bar{\bar{x}}}\right) \qquad \cdots \quad (8)$$

The average of the sample means $\overline{\overline{x}}$ and covariance matrices **S** are unbiased estimates of $\underline{\mu}$ and Σ, respectively when the process is in control. In this form, the procedure is usually called the Hotelling T_k^2 control chart. This is a directionally invariant control chart; that is, its ability to detect a shift in the mean vector only depends on the magnitude of the shift, and not in its direction. So the points plotted on the chart representing the T_k^2 values, which represents the vertical axis of the chart, (Wang, 2012).

The (construction stage) is to obtain an in-control set of observations so that control Limits can be established for (Stage of Use) which is the monitoring of future production control limits (Guoxi & Shing, 2008):

-Stage of Construction:

$$UCL_{T^2} = T^2 \sim \frac{2(m-1)(n-1)}{m(n-1)-1} F_{\alpha,2,m(n-1)-1} \quad \cdots \quad (9)$$
$$LCL_{T^2} = 0$$

Where m is number of samples and n is number of observations.

-Stage of Use:

$$UCL_{T^2} = T^2 \sim \frac{2(m+1)(n-1)}{m(n-1)-1} F_{\alpha,2,m(n-1)-1} \quad \cdots \quad (10)$$
$$LCL_{T^2} = 0$$

From (9) and (10) formulas, note that UCL_{T^2} in the stage of Construction and Use are unequal. These authors present tables indicating the recommended minimum value of *m* for sample sizes of n = 3, 5, and 10 and for *p* = 2, 3, 4, 5, 10, and 20 quality characteristics (in this research, p = 2). The recommended values of *m* are always greater than 20 preliminary samples.

2.2 F-Control Chart

Researcher suggested the use of F-Chart, through the three stages and as following:

1- Test of bivariate normality distribution for means of samples by using χ^2. We compute the Mahalanobis values, which must be less than tabulated χ^2 at level of significant $\alpha\%$ and degrees of freedom (p) but if were greater than the χ^2 value then these samples will be deleted from the analysis.

2- Test of hypothesis $H_0 : \Sigma = \Sigma_0$ for all samples, and $\Sigma_0 = S$ (from equations 6 and 7) by using S-Chart and as follows:

S-Chart is based on the sample generalized variance, |**S** |. This statistic, which is the determinant of the sample covariance matrix, is a widely used measure of multivariate dispersion. Where:

$$E(|S|) = b_1|\Sigma| \quad \cdots \quad (11)$$

And

$$V(|S|) = b_2|\Sigma|^2 \quad \cdots \quad (12)$$

Where

$$b_1 = \frac{1}{(n-1)^p}\prod_{i=1}^{p}(n-i)$$

And

$$b_2 = \frac{1}{(n-1)^{2p}}\prod_{i=1}^{p}(n-i)\left[\prod_{j=1}^{p}(n-j+2)-\prod_{j=1}^{p}(n-j)\right]$$

Therefore limits of the control chart for generalized variance would be:

$$\begin{aligned} UCL &= |\Sigma|\left(b_1 + 3b_2^{1/2}\right) \\ CL &= b_1|\Sigma| \qquad \cdots \quad (13) \\ LCL &= |\Sigma|\left(b_1 - 3b_2^{1/2}\right) \end{aligned}$$

If the points are out of control then these points will be deleted from the analysis.

3- Calculation of T_k^2 values from equation (8), so the points plotted on the chart representing the F_k values, which represents the vertical axis of the chart and with equation (9), we get the following formula:

$$F_k^c = \frac{m(n-1)-1}{2(m-1)(n-1)} n\left(\underline{\bar{x}}_k - \underline{\bar{\bar{x}}}\right)' S^{-1}\left(\underline{\bar{x}}_k - \underline{\bar{\bar{x}}}\right) \quad \cdots \quad (14)$$

Formula (11) is used only in the construction stage, but in the stage of use, the following formula is used:

$$F_k^u = \frac{m(n-1)-1}{2(m+1)(n-1)} n\left(\underline{\bar{x}}_k - \underline{\bar{\bar{x}}}\right)' S^{-1}\left(\underline{\bar{x}}_k - \underline{\bar{\bar{x}}}\right) \quad \cdots \quad (15)$$

The (construction stage) is to obtain an in-control set of observations so that control Limits can be established for (Stage of Use) which is the monitoring of future production control limits:

-Stage of Construction:

$$\begin{aligned} UCL_F &= F_{\alpha,2,m(n-1)-1} \qquad \cdots \quad (16) \\ LCL_F &= 0 \end{aligned}$$

-Stage of Use:

It can be obtained through the formula (16), which means that the control limits in both cases should be fixed.

3. Application

This side deals with the use of constructed bivariate F-Control Chart (Phase-1) Corresponding to T^2;

-Chart for quality control on measurable properties. For the purpose of obtaining data on the subject of the study, we collected the data from the Erbil laboratory and the data representing the quality properties of Yield Stress and Elongation (Strain) for steel product from factory (Erbil Steel).

The analysis assumes that there is a significant correlation between the study variables, the accuracy of the sampling and that the correlation matrix is not equal to the unit matrix, which was tested using the Kaiser-Meyer-Olkin (KMO) and Bartlett's Test of Sphericity through the SPSS which were summarized in the following tables:

Table 1: Correlation and the Test

Correlation	-.768
Sig. (1-tailed)	.000

Table 2: KMO and Bartlett's Test

Kaiser-Meyer-Olkin Measure of Sampling Adequacy.		.501
Bartlett's Test of Sphericity	Approx. Chi-Square	175.849
	df	1
	Sig.	.000

The (KMO) measure should be greater than 0.50 and is inadequate if less than 0.50. Here it is 0.501 which means it is not bad. The Bartlett test should be significant (i.e., a significance value of less than .05); this means that the variables are correlated to a high enough degree (p-value is less than .05, indicating that the correlation matrix is significantly different from an identity matrix). Computer packages Excel and MATLAB are designed for this purpose and to construct both quality control charts for the T^2-Chart and F-Chart**.**

3.1 T^2-Control Chart

Phase 1

A MATLAB language program is designed to calculate the table of control and chart it on T^2-Chart and based on the equations (9) and (10) for (m=40, n=5) phase (1). Through which T^2-Chart was obtained:

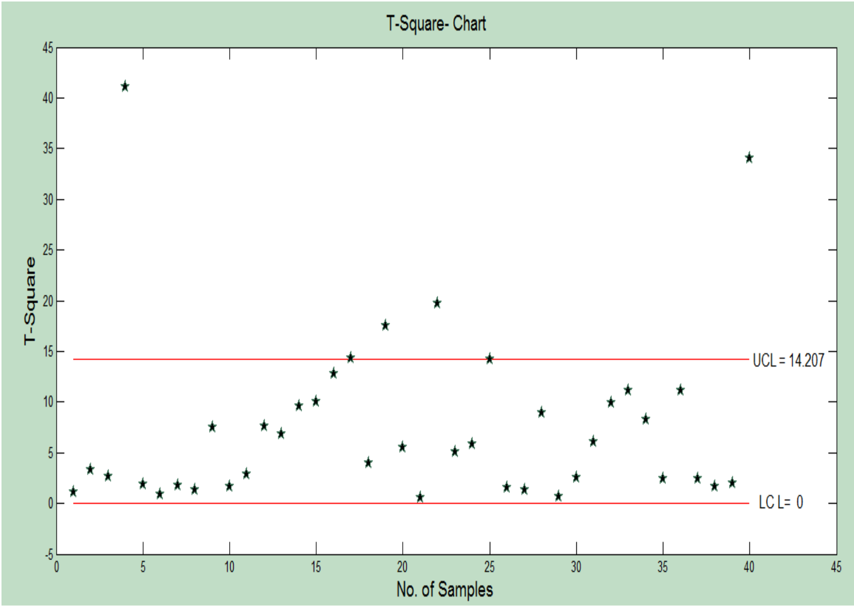

Figure 2: T^2–Chart

From figure (2) we observe that the process is out of control because all points are not located between (upper and lower) control limits. In this figure we observe that six points are out of control and above upper control limit.

In this type of charts we plot modified chart after removing six points which were located out of upper control limit. We obtained variance of data as in the following table:

Table 3: variance of data phase (1) for modified T^2 Control Chart

Var–Cov matrix	Total Variance	General Variance
$\begin{matrix} 3597.4 & -159.59 \\ -159.59 & 12.447 \end{matrix}$	3609.847	19307.87

Through which modified T^2-Chart for (m=34, n=5) was obtained.

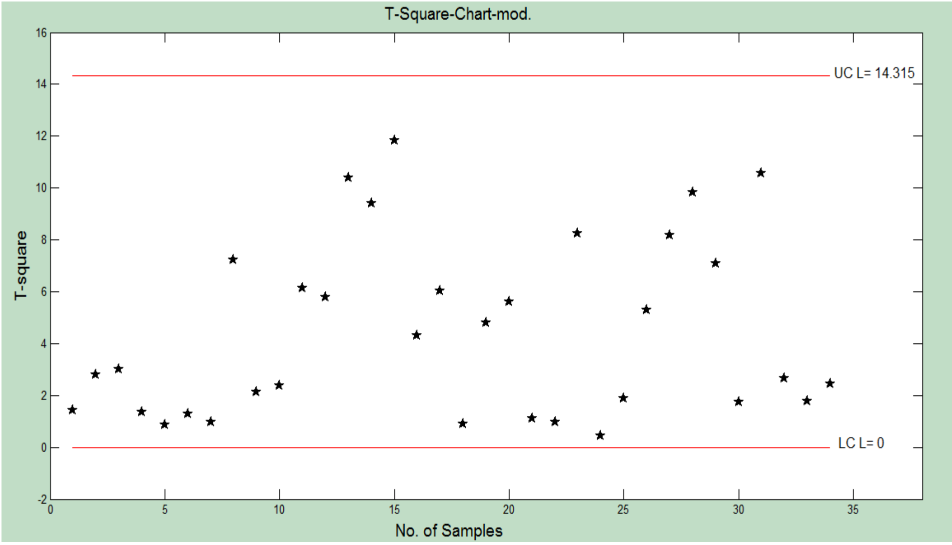

Figure 3: T^2–Chart-MOD

Figure (3) shows that all the points are within the limits of control. The data in Table (A1) for no. of samples is equal to 34 and it is appropriate in the composition of this chart, which means that this chart can be relied upon and used in the future for the same properties of quality.

Phase 2

The modified T^2–Chart in Figure (3) was used to control future data (new data) in Table (A2) for (m=25, n=5), it was as follows:

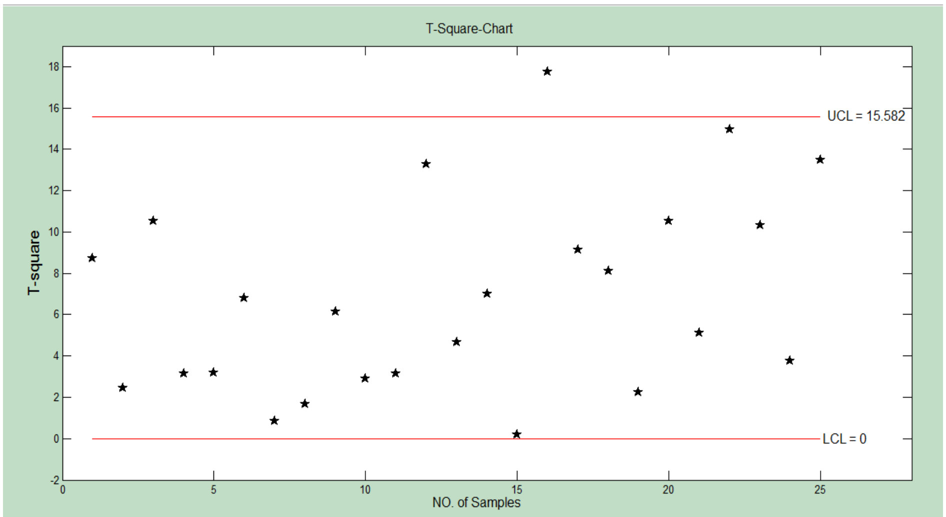

Figure 4: T^2– chart (Phase.2)

From figure (4) we show that the process is out of control because not all points are located between (upper and lower) control limits. In this figure we observe that one point is out of control and above upper control limit.

3.2 F-Control Chart

Phase 1

To construct the proposed F-Chart, we will follow these three steps:

1. Test the Bivariate normal distribution of the means of the samples summarized in Table (A1) using χ^2 test. We obtained the Mahalanobis values shown in Table (A3), which showed that the values (4 and 40) were greater than the χ^2 value at level of significant (5%) and degrees of freedom (2) which is equal to (5.99) so these two samples will be deleted from the analysis.

2. Construct S-Chart for the remaining (38) samples after deletion of samples (4 and 40) based on equation (13) and through a program designed for this purpose in MATLAB, the S-Chart is obtained as follows:

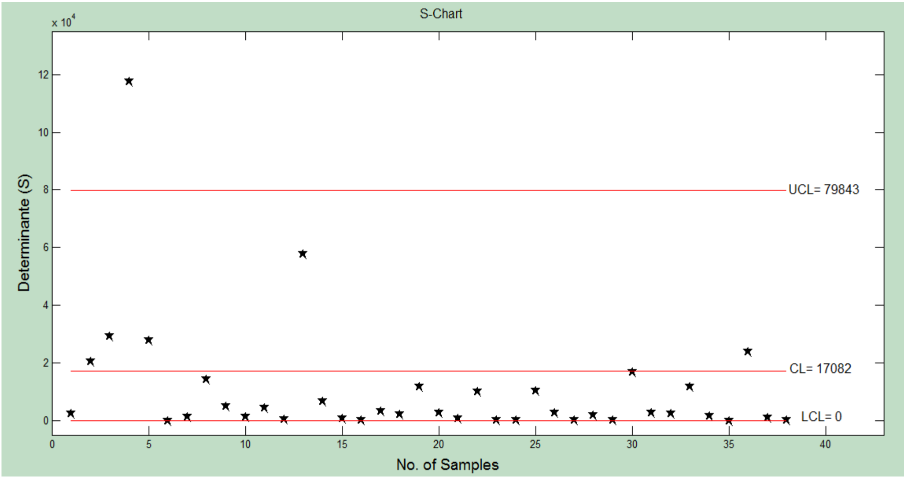

Figure 5: S– Chart

Through the S-Chart we note that there is one point outside the control limits which is sample (4) so it will be deleted and the construct of the following modified S-Chart will as in the following:

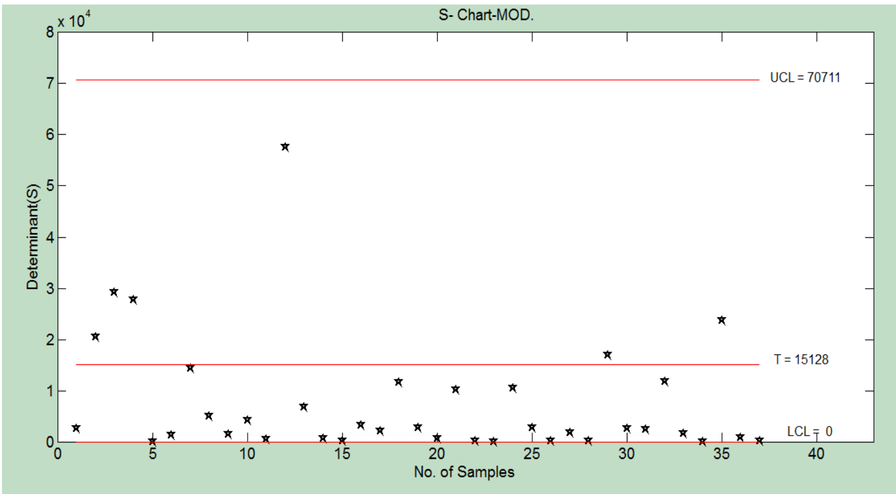

Figure 6: S– Chart-MOD

Note that all points drawn on the S-Chart are inside the limits of control so move to step three:

3- A MATLAB language program is designed to calculate the values F and construct F-Chart based on the equation (13), for (37) samples and n=5 as follows:

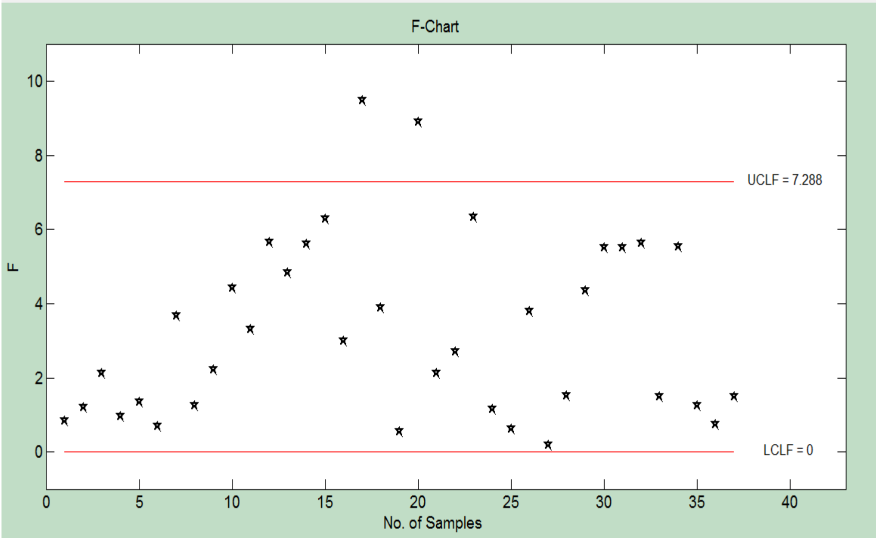

Figure 7: F–Chart

From Figure (7) we observe that the process is out of control because not all points are located between (upper and lower) control limits. In this figure we observe that two points are out of control (samples 17 and 20) and above upper control limit.

In this type of charts we plot modified chart after removing two points which were located out of upper control limit (for m=35, n=5). Through F-Chart we obtained:

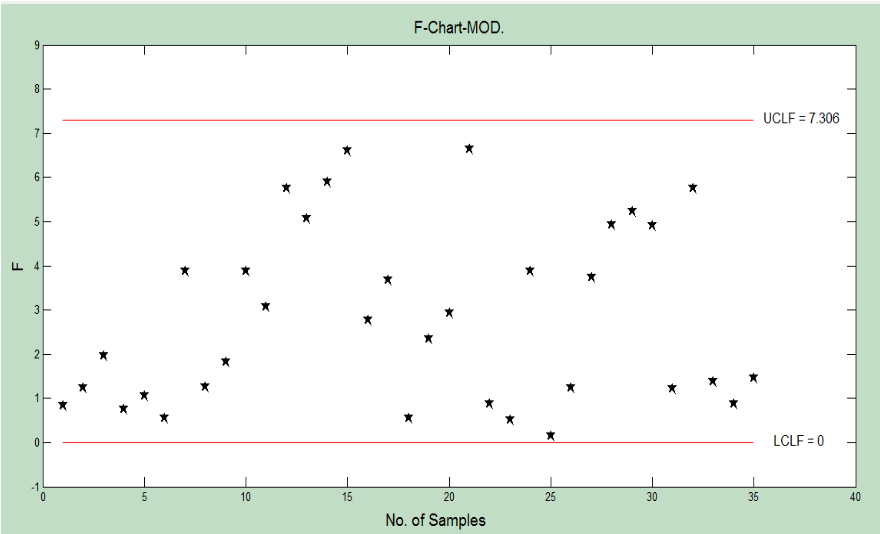

Figure 8: F– chart-MOD.

Figure (8) shows that all the points are within the limits of control and according to the composition of the chart for the first time, the data in Table (A1) for no. of samples is equal to 35 is appropriate in the composition of this chart, which means that this chart can be relied upon and used in the future for the same properties of quality.

We obtained variance of data as in the following table:

Table 4: Variance of data phase (1) for modified F-Control Chart

Var–Cov matrix	Total Variance	General Variance
3061.5 -133.47 -133.47 11.008	3072.5	15887

By comparing Table (3) and (4), we note that the General and Total variance of the proposed F-Chart is less than the T^2-Chart. Although the number of samples used in the proposed chart construction is equal to (35), it is greater than T^2-Chart which is equal (34).

Phase 2

The modified F–Chart in Figure (8) was used to control future data (new data) in Table (A2) for (m=25, n=5), it was as follows:

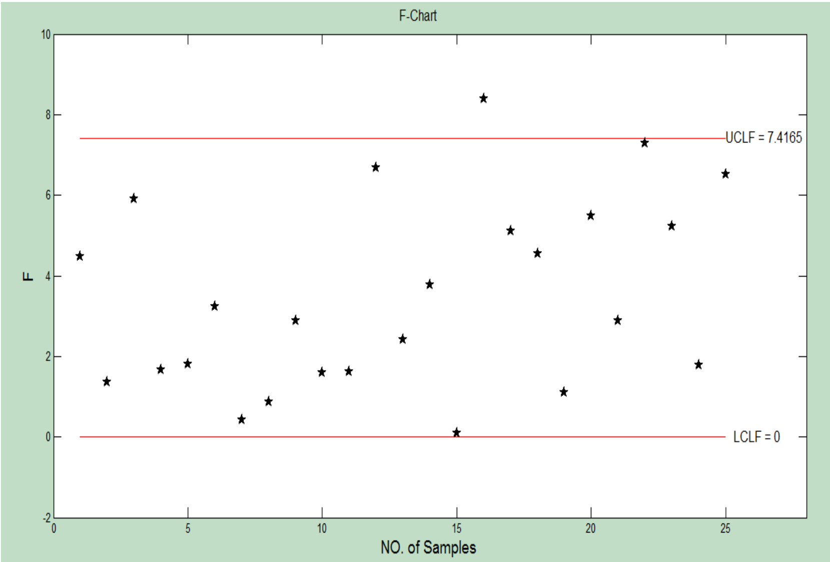

Figure 9: F– chart (Phase 2)

From figure (9) we show that the process is out of control because not all points are located between (upper and lower) control limits. In this figure we observe that one point is out of control and above upper control limit.

4. Discussion and Conclusion

Through this application side study, it was concluded that:

1- Appropriate data for the construction of $T^2$and F-chart for the first time (phase-1) in Erbil factory to control and monitoring of future production of iron.

2- We notice that the total and general variance of the proposed F-chart is less than that of the T^2-chart, which means that the F-chart is more accurate than the T^2-chart, although the number of samples used in the proposed chart construction is greater than T^2-Chart.

3- Through (phase-2) we observe that the process is out of control, because there is one point outside the limits of control which means that the product does not meet the required specifications in (phase-1) depending on the T^2-chart and F-chart.

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Appendix

Table A1: Data of phase-1

Sample	x_1	x_2	Sample	x_1	x_2	Sample	x_1	x_2	Sample	x_1	x_2
1	569.70	17.70	11	520.00	20.40	21	588.80	17.40	31	551.60	18.50
1	604.50	16.00	11	561.60	17.10	21	600.70	18.10	31	533.40	19.10
1	599.90	17.00	11	599.00	18.50	21	471.80	24.10	31	547.90	19.10
1	417.80	28.10	11	570.00	17.90	21	492.50	22.00	31	608.64	14.00
1	431.90	26.20	11	612.00	15.90	21	480.50	21.70	31	718.39	16.00
2	411.60	25.00	12	610.00	13.60	22	457.40	17.70	32	624.41	15.00
2	588.80	15.10	12	535.00	19.40	22	487.40	18.60	32	559.12	16.50
2	595.70	15.00	12	623.60	15.50	22	446.50	18.70	32	560.77	17.50
2	584.90	11.80	12	573.60	13.20	22	459.50	21.70	32	570.28	14.00
2	423.80	24.50	12	573.00	16.20	22	439.70	19.80	32	604.19	12.00
3	435.70	23.10	13	536.40	17.60	23	446.00	15.50	33	609.76	13.50
3	455.00	19.40	13	534.80	16.50	23	449.20	22.60	33	610.82	12.50
3	615.50	21.60	13	579.30	15.80	23	519.60	22.60	33	522.50	16.00
3	650.00	18.70	13	563.50	17.00	23	466.70	25.10	33	540.44	17.50
3	599.40	18.50	13	540.20	14.40	23	485.10	23.40	33	496.71	17.00
4	687.00	20.10	14	594.30	15.90	24	463.20	24.40	34	550.19	14.00
4	595.60	16.10	14	471.00	23.00	24	459.70	22.20	34	554.88	18.00
4	699.10	14.70	14	473.20	24.50	24	496.60	21.30	34	556.13	17.50
4	731.20	11.10	14	577.90	22.80	24	464.00	23.60	34	684.77	17.50
4	709.90	13.00	14	416.50	32.80	24	470.20	22.90	34	667.78	16.00
5	745.80	10.90	15	446.30	23.60	25	438.00	23.50	35	645.49	12.50
5	529.10	16.20	15	405.60	31.10	25	446.40	21.70	35	571.61	16.00
5	568.60	15.80	15	471.90	22.10	25	438.20	20.70	35	583.18	15.50
5	551.30	16.60	15	436.10	21.90	25	453.90	20.90	35	567.46	17.00
5	399.50	29.00	15	499.30	20.30	25	465.00	19.40	35	433.10	26.00

6	397.90	29.40	16	440.80	22.10	26	465.30	20.10	36	448.10	23.10
6	389.40	28.60	16	449.20	24.50	26	584.70	19.00	36	467.20	23.80
6	676.50	12.80	16	450.00	24.90	26	583.60	18.00	36	444.80	25.00
6	634.10	12.50	16	457.00	17.10	26	583.90	18.30	36	446.20	25.20
6	668.60	11.80	16	467.30	17.10	26	573.80	14.20	36	442.90	24.30
7	563.90	16.70	17	458.10	17.60	27	575.00	13.80	37	466.90	20.90
7	546.60	17.70	17	454.50	21.50	27	560.90	16.40	37	465.00	26.00
7	565.90	17.70	17	465.40	20.80	27	495.50	19.80	37	466.60	25.20
7	564.40	19.30	17	445.60	18.60	27	529.60	20.70	37	491.60	19.30
7	570.00	18.70	17	461.90	23.20	27	538.50	20.10	37	605.00	18.90
8	570.10	20.10	18	484.70	23.50	28	521.10	17.80	38	483.80	20.30
8	545.00	16.20	18	512.70	23.70	28	514.70	16.20	38	493.10	22.80
8	544.80	17.70	18	600.70	18.60	28	530.00	16.10	38	487.90	22.50
8	547.00	17.10	18	585.40	17.30	28	486.50	18.30	38	497.00	20.00
8	505.00	19.30	18	610.90	17.70	28	499.50	19.50	38	543.70	22.20
9	514.70	18.20	19	455.90	29.20	29	500.00	18.70	39	524.10	21.90
9	528.10	20.90	19	479.90	24.70	29	531.20	21.90	39	509.20	20.70
9	425.00	23.50	19	477.10	27.30	29	537.40	17.20	39	544.60	20.00
9	420.00	27.20	19	431.90	23.30	29	561.50	19.10	39	535.10	20.40
9	440.00	27.90	19	438.90	25.60	29	511.30	17.90	39	539.50	22.80
10	511.40	25.70	20	455.70	22.30	30	531.40	18.20	40	651.70	10.40
10	489.70	21.30	20	623.50	21.20	30	529.00	19.00	40	737.60	12.60
10	504.20	21.00	20	497.60	20.30	30	598.20	15.70	40	696.80	12.20
10	551.60	19.20	20	589.30	21.60	30	571.80	17.10	40	637.20	15.60
10	565.50	19.30	20	620.00	18.10	30	560.60	16.60	40	662.40	14.70

Table A2: Data of phase-2

Sample	x_1	x_2	Sample	x_1	x_2	Sample	x_1	x_2
1	675.00	15.30	11	611.10	16.60	21	484.30	23.60
1	608.60	17.00	11	596.00	20.60	21	471.50	25.10
1	614.20	19.10	11	584.40	19.60	21	491.30	22.60
1	605.50	22.30	11	569.20	18.70	21	476.60	24.10
1	570.60	18.20	11	579.80	19.00	21	472.40	27.20
2	646.00	17.10	12	430.00	16.70	22	466.50	25.40
2	558.40	15.20	12	486.50	15.30	22	469.60	21.50
2	556.10	17.00	12	463.50	14.00	22	492.20	21.90
2	606.00	18.60	12	466.20	18.60	22	489.20	21.70
2	542.70	18.80	12	470.00	21.10	22	436.80	26.50
3	451.10	24.80	13	280.00	20.50	23	503.10	25.40
3	453.20	24.40	13	504.30	17.80	23	452.80	22.60
3	474.20	25.80	13	450.90	17.00	23	594.10	18.50
3	498.50	23.80	13	467.00	23.70	23	560.60	22.50
3	487.10	25.40	13	355.00	18.40	23	571.00	19.80
4	493.00	21.51	14	471.00	18.10	24	615.00	17.10
4	634.70	14.80	14	522.10	19.00	24	603.50	17.50
4	591.80	14.60	14	458.00	23.20	24	650.00	17.20
4	618.80	16.00	14	442.30	21.40	24	639.40	18.60
4	540.40	18.40	14	446.50	23.70	24	620.20	18.34
5	569.40	17.90	15	424.20	20.60	25	626.10	19.04
5	535.50	19.60	15	438.50	16.90	25	607.10	17.60
5	627.60	15.60	15	426.30	19.60	25	575.60	15.70
5	600.50	16.40	15	403.20	32.10	25	582.70	14.10
5	633.10	16.70	15	401.90	31.20	25	639.50	16.90

6	543.60	21.10	16	401.70	29.30
6	540.50	24.70	16	413.00	29.10
6	549.30	22.60	16	404.90	29.30
6	556.30	20.90	16	405.90	27.30
6	558.20	22.60	16	450.00	25.10
7	582.60	21.10	17	455.90	26.70
7	543.40	21.00	17	451.40	27.20
7	530.70	22.20	17	453.00	19.60
7	536.10	21.40	17	644.60	16.10
7	527.90	18.70	17	474.00	15.50
8	540.70	19.00	18	472.20	20.70
8	548.90	16.10	18	486.20	20.60
8	580.60	12.50	18	487.30	19.70
8	573.50	18.30	18	425.50	24.40
8	579.60	23.40	18	519.70	25.50
9	592.50	17.50	19	512.20	21.80
9	576.00	18.30	19	590.60	15.80
9	573.80	19.30	19	555.80	18.50
9	473.10	13.10	19	602.70	16.90
9	458.20	20.90	19	418.20	29.10
10	472.30	22.50	20	407.70	31.00
10	592.90	17.30	20	409.60	32.20
10	626.60	13.90	20	639.70	18.60
10	606.80	16.00	20	647.20	19.80
10	608.40	15.80	20	686.80	16.60

Table A3: values Mahalanobis

sample	Mean(x_1)	Mean(x_2)	Mahalanobis
1	524.7600	21.0000	0.57652
2	520.9600	18.2800	0.91829
3	551.1200	20.2600	1.04545
4	684.5600	15.0000	8.61599
5	558.8600	17.7000	0.35615
6	553.3000	19.0200	0.23071
7	562.1600	18.0200	0.29945
8	542.3800	18.0800	0.31983
9	465.5600	23.5400	1.97736
10	524.4800	21.3000	0.86018
11	572.5200	17.9600	0.54280
12	583.0400	15.5800	1.83171
13	550.8400	16.2600	2.05641
14	506.5800	23.8000	3.98670
15	451.8400	23.8000	2.34359
16	452.8600	21.1400	2.79535
17	457.1000	20.3400	3.40801
18	558.8800	20.1600	1.42970
19	456.7400	26.0200	5.85554
20	557.2200	20.7000	2.08290
21	526.8600	20.6600	0.36891
22	458.1000	19.3000	5.22555
23	473.3200	21.8400	1.05543
24	470.7400	22.8800	1.42989
25	448.3000	21.2400	3.12720

26	558.2600	17.9200	0.27885
27	539.9000	18.1600	0.31990
28	510.3600	17.5800	2.73448
29	528.2800	18.9600	0.13377
30	558.2000	17.3200	0.57014
31	591.9860	17.3400	1.20235
32	583.7540	15.0000	2.60639
33	556.0460	15.3000	3.53160
34	602.7500	16.6000	1.58782
35	560.1680	17.4000	0.50077
36	449.8400	24.2800	2.77978
37	499.0200	22.0600	0.87759
38	501.1000	21.5600	0.53639
39	530.5000	21.1600	0.96977
40	677.1400	13.1000	6.63080