Classification of All Primitive Groups of Degrees Four and Five

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Abstract: Let X be a compact Riemann surface of genus g and $\mu: X \to \mathbb{P}^1$ be indecomposable meromorphic function of Riemann sphere $\mathbb{P}^1$ by X. Isomorphisms of such meromorphic functions are in one to one correspondence with conjugacy classes of r tuples $(x_1, x_2, \dots, x_r)$ of permutations in S_n such that $x_1 . x_2 \dots x_r = 1$ and
$G = < x_1, x_2, \dots, x_r >$ a subgroup of S_n.
Our goal of this work is to give a classification in the case where X is of genus 1 and the subgroup G is a primitive subgroup of S_4 or S_5. We present the ramification types for genus 1 to complete such a classification. Furthermore, we show that the subgroups D_{10} and C_5 of S_5 do not possesses primitive genus 1 systems.

Keywords*:* **Primitive Groups, Indecomposable Meromorphic Functions, Genus g Systems**

1. Introduction

Let R be a compact Riemann surface of genus g and that

$$\mu: R \to \mathbb{P}^1 \qquad (1)$$

is an indecomposable meromorphic function where $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. For every meromorphic function, there is a number n such that the fiber $\mu^{-1}(p)$ is of n for all but finitely many points $p \in \mathbb{P}^1$. The number n is called the degree of μ. The points b where $\mu^{-1}(b) < n$ are called the branch points of μ. Let $B \leq \mathbb{P}^1$ be the set of branch points of μ. It is well known that B is a finite set. So one can label the points in B by $\{b_1, \dots, b_r\}$. For any $p \in \mathbb{P}^1 \backslash B$, the fundamental group $\pi_1(\mathbb{P}^1 \backslash B, p)$ acts on $\mu^{-1}(p)$ via path lifting. It gives a group homomorphism $\rho: \pi_1(\mathbb{P}^1 \backslash B, p) \to S_n$. The image of ρ is called the monodromy group of μ and denoted by Mon(R, μ). If R is connected, then Mon(R, μ) is a transitive subgroup of S_n. Furthermore $\pi_1(\mathbb{P}^1 \backslash B, p)$ is generated by all homotopy classes of loops γ_i winding once around the point b_i. The loops γ_i can be chosen so that the generators γ_i satisfy the only relation

$$\gamma_1 \cdot \gamma_2 \cdot \dots \gamma_r = 1. \qquad (2)$$

Applying ρ to the canonical generators of $\pi_1(\mathbb{P}^1 \backslash B, p)$ gives the generators of a product one generating tuple in Mon(R, μ). We set, $x_i = \rho(\gamma_i)$, $1 \leq i \leq r$ and $G = \text{Mon}(R, \mu)$, then the following statements are true:

$$G = \langle x_1, x_2, \dots, x_r \rangle \quad (3)$$

$$\prod_1^r x_i = 1, x_i \in G^{\#}, \ i = 1, \dots, r. \quad (4)$$

$$\sum_{i=1}^r ind\, x_i = 2(n + g - 1), \quad (5)$$

Where $\mathrm{ind} x_i$ is the minimal number of transpositions needed to express x_i as a product. Equation (5) is known as the Riemann-Hurwitz formula. Let C_i be the conjugacy class of x_i. Then the multi-set of non trivial conjugacy classes $C = \{C_1, \dots, C_r\}$ in G is called the ramification type of the cover μ. While x_i is not uniquely determined by R and μ, the class C_i is uniquely determined by R and μ. This non-uniqueness will be a very interesting fact that allows us to discuss braid actions (Salih, 2014).

A transitive subgroup $G \leq S_n$ is a genus g group if there exist $x_1, x_2, \dots, x_r \in G$ satisfying (3), (4) and (5) above, and we call $(x_1, x_2, \dots, x_r)$ a genus g system of G. If the action of G on $\{1, \dots, n\}$ is primitive, we call G a primitive genus g group and $(x_1, x_2, \dots, x_r)$ a primitive genus g system. Our goal is to classify primitive genus one systems up to diagonal conjugation and braiding for degree $n = 4$ and 5. We achieve this classification with the aid of the computer algebra system GAP.

2. Nielsen Classes and Hurwitz Spaces

Definition 2.1 (Vӧlklein, 1996).

Let $C_1, \dots, C_r$ be a non-trivial conjugacy classes of a finite group G. The set of generating systems $(x_1, \dots, x_r)$ of G with $x_1 \dots x_r = 1$ and such that there is a permutation $\pi \in S_r$ with $x_i \in S_{\pi(i)}$ for $i = 1, \dots, r$ is called a Nielsen class and it denoted by $\mathcal{N}(C)$, where $C = (C_1, \dots, C_r)$.

Each Nielsen class is the disjoint union of braid orbits, which are defined as the smallest subsets of the Nielsen class closed under the braid operations (Vӧlklein, 1996).

$$(x_1, \dots, x_r)^{Q_i} = \left(x_1, \dots, x_{i+1}, x_{i+1}^{-1} x_i x_{i+1}, \dots, x_r\right) \quad (6)$$

for $i = 1, \dots, r$.

We denote by O_r, the space of subsets of $\mathbb{C}$ of cardinality r.

Definition 2.2 (Vӧlklein, 1996).

Let $B \in O_r$ and $b_0 \in \mathbb{P}^1 \setminus B$, we call a map $\varphi: \pi_1(\mathbb{P}^1 \setminus B, b_0) \to G$ admissible if it is a surjective homomorphsim, and $\varphi(\theta_b) \neq 1$ for each $b \in B$. Here θ_b is the conjugacy class of $\pi_1(\mathbb{P}^1 \setminus B, b_0)$.

Definition 2.3 (Vӧlklein, 1996).

Let $B \in O_r$ and $\varphi: \pi_1(\mathbb{P}^1 \setminus B, \infty) \to G$ be admissible. Then we say that two pairs (B, φ) and $(\bar{B}, \bar{\varphi})$ are $\mathcal{A}$-equivalent if and only if $B = \bar{B}$ and $\bar{\varphi} = a \circ \varphi$ for some $a \in \mathcal{A}$.

Definition 2.4 (Vӧlklein, 1996).

Let $[B, \varphi]_{\mathcal{A}}$ denote the $\mathcal{A}$-equivalence class of (B, φ). The set of equivalence classes $[B, \varphi]_{\mathcal{A}}$ is denoted by $\mathcal{H}_r^{\mathcal{A}}(G)$ and is called the Hurwitz space of G-covers.

To define the topology of the Hurwitz space $\mathcal{H}_r^{\mathcal{A}}(G)$, we assume that $B = \{b_1, \dots, b_r\} \in O_r$ and $D_1, \dots, D_r$ be distinct discs of $b_1, \dots, b_r$. A neighborhood of $[B, \varphi]_{\mathcal{A}}$ is the set of all $[\bar{B}, \bar{\varphi}]_{\mathcal{A}}$where $\bar{B} = \{b_1, \dots, b_r\}$ such that $b_i \in D_i$ for $i = 1, \dots, r$ and $\bar{\varphi}$ is the composition of φ with the canonical isomorphisms

$$\pi_1(\mathbb{P}^1 \setminus \bar{B}, \infty) \to \pi_1(\mathbb{P}^1 \setminus \{D_1 \cup \dots \cup D_r\}, \infty) \to \pi_1(\mathbb{P}^1 \setminus B, \infty)$$

This gives a topology on $\mathcal{H}_r^{\mathcal{A}}(G)$.

We define $\mathcal{E}_r(G) = \{(x_1, \dots, x_r): G = \langle x_1, \dots, x_r \rangle, \prod_{i=1}^{r} x_i = 1, x_i \in G^{\#}, i = 1,2, \dots, r\}$. Let $\mathcal{A} \leq Aut(G)$. Then the subgroup $\mathcal{A}$ acts on $\mathcal{E}_r(G)$ via sending $(x_1, \dots, x_r)$ to $(a(x_1), \dots, a(x_r))$, for $a \in \mathcal{A}$, which is known as the diagonal conjugation. This action commutes with the operations (6). Thus $\mathcal{A}$ permutes the braid orbits. If $\mathcal{A} = Inn(G)$, then it leaves each braid orbit invariant (Völklein, 1996). Let $\mathcal{E}_r^{in}(G) = \mathcal{E}_r(G)/Inn(G)$.

Lemma 2.5 (Völklein, 1996)**.** The map $\Psi_{\mathcal{A}}: \mathcal{H}_r^{\mathcal{A}}(G) \to O_r$, $\Psi_{\mathcal{A}}([P, \varphi]) = P$ is *covering*.

The topology on the Hurwitz space $\mathcal{H}_r^{\mathcal{A}}(G)$ completely determined by the action of the fundamental group $\pi_1(O_r, P_0)$ where $P_0 = \{1, \dots, r\}$ is the base point in O_r via path lifting. To set out this action, we need a parameterization $\Psi_{\mathcal{A}}^{-1}(P_0)$. The fiber $\Psi_{\mathcal{A}}^{-1}(P_0) = \{[P_0, \varphi]_{\mathcal{A}}: \varphi: \pi_1(\mathbb{P}^1 \setminus B, \infty) \to G$ is admissible$\}$. This φ gives a product one generating tuple $(x_1, \dots, x_r)$ of G.

Lemma 2.6 (Völklein, 1996). We obtain a bijection $\Psi_{\mathcal{A}}^{-1}(P_0) \to \mathcal{E}_r^{\mathcal{A}}(G)$ by sending $[P_0, \varphi]_{\mathcal{A}}$ to the generators $(x_1, \dots, x_r)$ where $x_i = \varphi([\gamma_i])$ for $i = 1, \dots, r$.

The image $\mathcal{N}^{\mathcal{A}}(C)$ of $\mathcal{N}(C)$ is the union of braid orbits. If $\Psi_{\mathcal{A}}$ in Lemma 2.5. restricts to a connected component $\mathcal{H}$ of $\mathcal{H}_r^{\mathcal{A}}(G)$, then Lemma 2.6. implies that the fiber in $\mathcal{H}$ over P_0 corresponds to the set $\mathcal{N}^{\mathcal{A}}(C)$. As a result, we have the following proposition

Proposition 2.7 (Gehao, 2011). There is a one-to-one correspondence between connected components of $\mathcal{H}_r^{in}(C)$ and braid orbits on Nielsen classes $\mathcal{N}(C)$. In particular, $\mathcal{H}_r^{in}(C)$ is connected if and only if there is only braid orbit on $\mathcal{N}(C)$.

Definition 2.8 (Völklein, 1996)**.** Two generating tuples are braid equivalent if they lie in the same orbit under the group generated by the braid action and diagonal conjugation by $Inn(G)$.

Theorem 2.9 (Völklein, 1996). Two generating tuples are braid equivalent if and only if their corresponding covers are equivalent.

Few results were known about the Hurwitz spaces $\mathcal{H}_r^{in}(C)$. For instance, Clebsch (1872) shows that if $G = S_n$ and let $C = (C, \dots, C)$ be r-tuple consisting of r copies the class C of transpositions, then the corresponding Hurwitz space $\mathcal{H}_r^{in}(C)$ is connected. Liu and Osserman (2008) generalized this result as follows. If $G = S_n$ and C_i represented by x_i where x_i is a single cycle of length $|x_i|$, then $\mathcal{H}_r(C)$ is connected. Furthermore, Fried (2006) shows that if $G = A_n, g > 0$, and all C_i are represented by 3-cylces then $\mathcal{H}_r(C)$, has one component if $g = g(X \backslash G) = 0$, and otherwise it has two components. James, Magaard and Shpectorov (2012) determined all braid orbits on Nielsen classes of primitive genus zero systems for A_5 and A_6. Recently, Salih and Akray (2016) determined all connected components for genus zero systems for A_8. Also, he classified all primitive groups of degree 4,5 and 6 for genus zero systems and show that the Hurwitz spaces for these groups are connected if $r \geq 3, n = 4$ and $r \geq 4, n = 5$ and $r \geq 5, n = 6$.

3. Methodology for Finding Ramification Types and Braid Orbits

A. We extract all primitive permutation groups G by using the GAP function All Primitive Groups (Degree Operation, n).

B. For given degrees 4,5,6 genus 1 and G, we compute all possible ramification types satisfying the Riemann-Hurwitz formula as follows.

Now, we discuss the computation of the indices, we give some alternative formula to compute index of an element in a group. Let G be a group acting on a finite set Ω and $|\Omega| = n$. If $x \in G$, define the index of x by $ind\ x = n - orb(x)$ (7) where $orb(x)$ is the number of orbits of $< x >$ on Ω. Also, $Fix\ x = \{w \in \Omega\ |xw = w\}$, $f(x) = |Fix\ x|$. Furthermore, $orb(x) = \frac{1}{d}\sum_{i=0}^{d-1} f(x^i)$, where x has order d.

Alternating group A_7.

Maximal subgroups

Order	Index	Structure	$G.2$	Character
12	5	A_4	$:S_4$	$1a + 4a$
10	6	D_{10}	$5:4$	$1a + 5a$
6	10	S_3	$2 \times S_3$	$1a + 4a + 5a$

		60	4	3	5	5			6	2	3
	p	power	A	A	A	A			A	A	AB
	p'	power	A	A	A	A			A	A	AB
	ind	$1A$	$2A$	$3A$	$5A$	B^*	fus	ind	$2B$	$4A$	$6A$
$\mathcal{X}_1$	+	1	1	1	1	1	:	++	1	1	1
$\mathcal{X}_2$	+	3	−1	0	b_5	*	|	+	0	0	0
$\mathcal{X}_3$	+	3	−1	0	*	b_5					
$\mathcal{X}_4$	+	4	0	1	−1	−1	:	++	2	0	−1
$\mathcal{X}_5$	+	5	1	−1	0	0	:	++	1	−1	1

Note that the above is a part of the character table of A_5. From the character table of A_5, we see that A_5 has elements of order 2,3 and 5. First, we compute fix points of elements x, which are equal to $1a + 4a = \mathcal{X}_1 + \mathcal{X}_4$ of the given orders. Second, we use formula (7), we obtain the following:
If x is an element of order 2, then

$$ind\ x = 5 - \frac{1}{2}[f(x^0) + f(x)] = 5 - \frac{1}{2}[5 + 1] = 2.$$

If x is an element of order 3, then

$$ind\ x = 5 - \frac{1}{3}[f(x^0) + f(x) + f(x^2)] = 5 - \frac{1}{3}[5 + 2 + 2] = 2.$$

If x is an element of order 5, then

$$ind\ x = 5 - \frac{1}{5}[f(x^0) + f(x) + f(x^2) + f(x^3) + f(x^4)] = 5 - \frac{1}{5}[5 + 0 + 0 + 0 + 0] = 4.$$

We can do the same steps for the other groups.

C. Compute the character table of G and remove those types which have zero structure constant. The structure constant can be computed by the following formula

$$n(C_1, \dots, C_k) = \frac{|C_1||C_2|\dots|C_k|}{|G|} \sum_{\chi \in Irr(G)} \frac{\chi(x_1)\chi(x_2)\dots\chi(x_k)}{\chi(1)^{k-2}} \quad (8)$$

With equation (8), we compute the number of k-tuples $(x_1, \dots, x_k)$ of elements x_i in the conjugacy class C_i of a group G such that $x_1 x_2 \dots x_k = 1$.

D. For each of the remaining ramification types, we use MAPCLASS package to compute braid orbits. As we know that the MAPCLASS package of James, Magaard, Shpectorov (2012) and Volklein (1996) is designed to perform braid orbit computations for a given finite group and given type.

4. Results

Here, we present our main results.

Lemma 4.1

The groups C_5 and D_{10} do not posses primitive genus one systems.

Proof. For C_5, it follows that each element in C_5 fix at most one point. So $ind\ x = 4$. From Riemann Huwritz formula, we have $10 = 2(5-1-1) = \sum_{i=1}^{r} ind\ x_i$. But no tuples satisfy this formula. Thus C_5 does not possess primitive genus one systems. The computations show that 5 ramification types passes formulas (3) and (4) as follows: $(2A, 2A, 2A, 2A, 2A), (2A, 2A, 2A, 5A)$, $(2A, 5A, 5A), (2A, 5B, 5B)$ and $(2A, 5A, 5B)$. But none of them passes formula (5). Therefore, D_{10} does not possess primitive genus one systems.

Lemma 4.2

The Hurwitz spaces, $\mathcal{H}_r^{in}(C)$ are connected if $r \geq 5$ and $n = 4$.

Proof. Since we have only one braid orbit for $r \geq 5$ and $n = 4$. From Proposition 2.7, it follows that $\mathcal{H}_r^{in}(C)$ are connected.

Corollary 4.3

For the group S_4, the Hurwitz spaces, $\mathcal{H}_r^{in}(C)$ are connected.

Proof. Since we have only one braid orbit for S_4. From Proposition 2.7, it follows that $\mathcal{H}_r^{in}(C)$ are connected.

Lemma 4.4

The Hurwitz spaces, $\mathcal{H}_r^{in}(C)$ are connected if $r \geq 6$ and $n = 5$.

Proof. The proof is similar as Lemma 4.2.

Corollary 4.5

For the group $AGL(1,5)$, The Hurwitz spaces, $\mathcal{H}_r^{in}(C)$ are connected.

Proof. The proof is similar as Lemma 4.2.

Lemma 4.6

The Hurwitz spaces, $\mathcal{H}_r^{in}(C)$ are disconnected if $r \geq 3$ and $n = 6$.

Proof. Since we have two or more braid orbits for some types C. From Proposition 2.7, it follows that $\mathcal{H}_r^{in}(C)$ are disconnected.

Table 1: Part 1: Primitive genus one systems of degree 5

Group	Ramification type	Number of orbits	Largest length
S_5	$(2A,2A,2A,2A,3A,3A,3A)$	1	8983
	$(2A,2A,2A,2A,2B,3A,3A)$	1	7432
	$(2A,2A,2A,2A,2B,2B,3A)$	1	5856
	$(2A,2A,2A,2A,2B,2B,2B)$	1	4480
	$(2A,2A,2A,2A,2A,3A,4A)$	1	7040
	$(2A,2A,2A,2A,2A,3A,6A)$	1	5325
	$(2A,2A,2A,2A,2A,2B,4A)$	1	5760
	$(2A,2A,2A,2A,2A,2B,6A)$	1	4080
	$(2A,2A,2A,2A,2A,2A,5A)$	1	3125
	$(2A,2A,2A,2A,2A,2A,3A,3A)$	1	45990
	$(2A,2A,2A,2A,2A,2A,2B,3A)$	1	37440
	$(2A,2A,2A,2A,2A,2A,2B,2B)$	1	29280
	$(2A,2A,2A,2A,2A,2A,2A,4A)$	1	35840
	$(2A,2A,2A,2A,2A,2A,2A,6A)$	1	26460
	$(2A,2A,2A,2A,2A,2A,2A,2A,3A)$	1	234360
	$(2A,2A,2A,2A,2A,2A,2A,2A,2B)$	1	188160
	$(2A,2A,2A,2A,2A,2A,2A,2A,2A,2A)$	1	1189440

Table 1: Part 2: Primitive genus one systems of degree 5

Group	Ramification Type	Number of orbits	Largest length	Ramification Type	Number of orbits	Largest length
S_5	$(4A,5A,6A)$	2	1	$(4A,4A,5A)$	1	1
	$(3A,3A,4A,4A)$	2	24	$(5A,6A,6A)$	1	1
	$(3A,3A,4A,6A)$	2	16	$(2B,3A,4A,4A)$	1	33
	$(3A,3A,6A,6A)$	2	14	$(2B,3A,4A,6A)$	1	26
	$(2B,3A,6A,6A)$	1	16	$(2B,2B,4A,4A)$	1	16
	$(2B,2B,4A,6A)$	1	18	$(2B,2B,6A,6A)$	1	12
	$(2A,4A,4A,4A)$	1	32	$(2A,4A,4A,6A)$	1	26

	$(2A,4A,6A,6A)$	1	16	$(2A,6A,6A,6A)$	1	10
	$(2A,3A,4A,5A)$	1	20	$(2A,3A,5A,6A)$	1	15
	$(2A,2B,4A,5A)$	1	15	$(2A,2B,5A,6A)$	1	10
	$(2A,2A,5A,5A)$	1	10	$(2A,2B,3A,3A,4A)$	1	224
	$(2A,3A,3A,3A,4A)$	1	272	$(2A,3A,3A,3A,6A)$	1	213
	$(2A,2B,3A,3A,6A)$	1	166	$(2A,2B,2B,3A,4A)$	1	180
	$(2A,2B,2B,3A,6A)$	1	123	$(2A,2B,2B,2B,4A)$	1	136
	$(2A,2B,2B,2B,6A)$	1	90	$(2A,2A,3A,4A,4A)$	1	212
	$(2A,2A,3A,4A,6A)$	1	168	$(2A,2A,3A,6A,6A)$	1	115
	$(2A,2A,3A,3A,5A)$	1	125	$(2A,2A,2B,2B,5A)$	1	75
	$(2A,2A,2B,4A,4A)$	1	176	$(2A,2A,2B,4A,6A)$	1	134
	$(2A,2A,2B,6A,6A)$	1	82	$(2A,2A,2B,3A,5A)$	1	100
	$(2B,2B,2B,4A,5A)$	1	100	$(2B,2B,2B,5A,6A)$	1	75
	$(2A,2A,3A,3A,3A,3A)$	1	1752	$(2A,2A,2B,3A,3A,3A)$	1	1468
	$(2A,2A,2B,2B,3A,3A)$	1	1170	$(2A,2A,2B,2B,2B,3A)$	1	900
	$(2A,2A,2B,2B,2B,2B)$	1	672	$(2A,2A,2A,3A,3A,4A)$	1	1376
	$(2A,2A,2A,3A,3A,6A)$	1	1072	$(2A,2A,2A,2B,3A,4A)$	1	1144
	$(2A,2A,2A,2B,3A,6A)$	1	820	$(2A,2A,2A,2B,2B,6A)$	1	612
	$(2A,2A,2A,2A,4A,4A)$	1	1088	$(2A,2A,2A,2A,4A,6A)$	1	832
	$(2A,2A,2A,2A,6A,6A)$	1	576	$(2A,2A,2A,2A,3A,5A)$	1	625
	$(2A,2A,2A,2A,2B,5A)$	1	500			

Table 1: Part 3. Primitive genus one systems of degree 5

Group	Ramification Type	Number of orbits	Largest length	Ramification Type	Numbe r of orbits	Largest Length
A_5	$(3A,3A,3A,5B)$	2	15	$(3A,5A,5B)$	1	1
	$(3A,5B,5B)$	1	1	$(2A,5B,5A)$	1	1
	$(3A,3A,3A,5A)$	2	15	$(3A,5A,5A)$	1	1
	$(2A,3A,3A,5A)$	1	20	$(2A,3A,3A,5B)$	1	20
	$(2A,2A,3A,5A)$	1	15	$(2A,2A,3A,5B)$	1	15
	$(2A,2A,2A,5A)$	1	10	$(2A,2A,2A,5A)$	1	10
	$(3A,3A,3A,3A,3A)$	2	432	$(2A,3A,3A,3A,3A)$	1	576
	$(2A,2A,3A,3A,3A)$	1	468	$(2A,2A,2A,3A,3A)$	1	360
	$(2A,2A,2A,2A,3A)$	1	270	$(2A,2A,2A,2A,2A)$	1	192
$AGL(1,5)$	$(2A,2A,4A,4B)$	1	6			
	$(4A,4B,5A)$	1	1			

Table 2: Primitive genus one systems of degree 4

Group	Ramification Type	Number of orbits	Largest length	Ramification Type	Number of orbits	Largest length
A_4	$(3A,3A,3B,3B)$	2	3	$(2A,3B,3B,3B)$	1	4

	$(2A, 2A, 3A, 3B)$	1	3	$(2A, 3A, 3A, 3A)$	1	4
S_4	$(3A, 4A, 4A)$	1	1	$(2A, 3A, 3A, 4A)$	1	8
	$(2B, 2A, 3A, 4A)$	1	3	$(2B, 2B, 4A, 4A)$	1	4
	$(2B, 2B, 3A, 3A, 3A)$	1	60	$(2B, 2B, 2B, 2A, 3A)$	1	24
	$(2B, 2B, 2A, 2A, 3A)$	1	9	$(2B, 2B, 2B, 3A, 4A)$	1	36
	$(2B, 2B, 2B, 2A, 4A)$	1	12	$(2B, 2B, 2B, 2B, 3A, 3A)$	1	270
	$(2B, 2B, 2B, 2B, 2A, 3A)$	1	108	$(2B, 2B, 2B, 2B, 2A, 2A)$	1	36
	$(2B, 2B, 2B, 2B, 2B, 4A)$	1	160			
	$(2B, 2B, 2B, 2B, ,2B, 2B, 3A)$	1	1215			
	$(2B, 2B, 2B, 2B, ,2B, 2B, 2A)$	1	480			
	$(2B, 2B, 2B, 2B, ,2B, 2B, 2B, 2B)$	1	5460			

Table 3: Primitive genus one systems of degree 6

Group	Ramification Type	Number of orbits	Largest length	Ramification Type	Number of orbits	Largest length
	$(5B, 5B, 5B)$	2	1	$(5A, 5B, 5B)$	2	1
	$(5A, 5A, 5B)$	2	1	$(5A, 5A, 5A)$	2	1
	$(4A, 5B, 5B)$	4	1	$(4A, 5A, 5A)$	4	1
	$(4A, 5A, 5A)$	4	1	$(4A, 4A, 5B)$	3	1
	$(4A, 4A, 5A)$	3	1	$(4A, 4A, 4A)$	4	1
	$(3B, 5A, 5B)$	1	1	$(3B, 4A, 5B)$	2	1
	$(3B, 4A, 5A)$	2	1	$(3A, 3A, 5B, 5B)$	2	30
	$(3A, 3A, 5A, 5A)$	2	30	$(3A, 3A, 5A, 5B)$	3	30
	$(3A, 3A4A, 5B)$	2	40	$(3A, 3A, 4A, 5A)$	2	40
	$(3A, 3A, 4A, 4A)$	3	48	$(3A, 3A, 3B, 5B)$	2	20
	$(3A, 3A, 3B, 5A)$	2	20	$(3A, 3A, 3B, 4A)$	2	24
	$(3A, 3A, 3B, 3B)$	2	18	$(2A, 3A, 5B, 5B)$	1	60
	$(2A, 3A, 5A, 5B)$	1	60	$(2A, 3A, 5A, 5A)$	1	60
	$(2A, 3A, 4A, 5B)$	1	90	$(2A, 3A, 4A, 5A)$	1	90
	$(2A, 3A, 4A, 4A)$	1	96	$(2A, 3A, 3B, 5B)$	1	40
	$(2A, 3A, 3B, 5A)$	1	40	$(2A, 3A, 3B, 4A)$	1	58
A_6	$(2A, 3A, 3B, 3B)$	1	20	$(2A, 2A, 5B, 5B)$	3	30
	$(2A, 2A, 5A, 5B)$	3	30	$(2A, 2A, 5A, 5A)$	3	30
	$(2A, 2A, 4A, 5B)$	3	40	$(2A, 2A, 4A, 5A)$	3	40
	$(2A, 2A, 4A, 4A)$	3	40	$(2A, 2A, 3B, 5B)$	1	30
	$(2A, 2A, 3B, 5A)$	1	30	$(2A, 2A, 3B, 4A)$	1	48
	$(2A, 2A, 3A, 3A, 5B)$	1	1500	$(2A, 2A, 3A, 3A, 5A)$	1	1500
	$(2A, 2A, 3A, 3A, 4A)$	1	2112	$(2A, 2A, 3A, 3A, 3B)$	1	972
	$(2A, 2A, 2A, 3A, 5B)$	1	1800	$(2A, 2A, 2A, 3A, 5A)$	1	1800
	$(2A, 2A, 2A, 3A, 4A)$	1	2448	$(2A, 2A, 2A, 3A, 3B)$	1	1080
	$(2A, 2A, 2A, 2A, 5B)$	3	675	$(2A, 2A, 2A, 2A, 5A)$	3	675
	$(2A, 2A, 2A, 2A, 4A)$	3	864	$(2A, 2A, 2A, 2A, 3B)$	1	972
	$(2A, 3A, 3A, 3A, 5B)$	1	1200	$(2A, 3A, 3A, 3A, 5A)$	1	1200
	$(2A, 3A, 3A, 3A, 4A)$	1	1824	$(2A, 3A, 3A, 3A, 3B)$	1	816
	$(3A, 3A, 3A, 3A, 5B)$	2	600	$(3A, 3A, 3A, 3A, 5A)$	2	600
	$(3A, 3A, 3A, 3A, 4A)$	2	768	$(3A, 3A, 3A, 3A, 3B)$	2	432
	$(3A, 3A, 3A, 3A, 3A, 3A)$	2	11880	$(2A, 3A, 3A, 3A, 3A, 3A)$	1	24320

	$(2A,2A,3A,3A,3A,3A)$	1	30672	$(2A,2A,2A,3A,3A,3A)$	1	37440
	$(2A,2A,2A,2A,3A,3A)$	1	44496	$(2A,2A,2A,2A,2A,3A)$	1	51840
	$(2A,2A,2A,2A,2A,2A)$	3	19440			
$PSL(2,5)$	$(3A,3A,5A)$	1	1	$(3A,5A,5A)$	1	1
	$(5A,5A,5A)$	1	1	$(3A,3A,5B)$	1	1
	$(3A,5A,5B)$	1	1	$(3A,5B,5B)$	1	1
	$(5B,5B,5B)$	1	1	$(2A,2A,3A,3A)$	1	18
	$(2A,2A,3A,5A)$	1	15	$(2A,2A,5A,5A)$	1	10
	$(2A,2A,3A,5B)$	1	15	$(2A,2A,5B,5A)$	1	5
	$(2A,2A,5B,5B)$	1	10	$(2A,2A,2A,2A,3A)$	1	270
	$(2A,2A,2A,2A,5A)$	1	150	$(2A,2A,2A,2A,5B)$	1	150
	$(2A,2A,2A,2A,2A)$	1	2880			
$PGL(2,5)$	$(2A,5A,6A)$	1	1	$(2B,6A,6A)$	1	1
	$(3A,4A,6A)$	2	1	$(4A,5A,6A)$	2	1
	$(2A,2A,2B,5A)$	1	5	$(2A,2A,2B,6A)$	1	6
	$(2B,2B,4A,6A)$	1	18	$(2B,4A,2A,3A)$	1	10
	$(2A,2B,4A,5A)$	1	15	$(2A,2A,2B,3A)$	1	33
	$(2A,2A,2B,5A)$	1	35	$(2A,2A,4A,4A)$	1	8
	$(2A,4A,4A,4A)$	1	32	$(4A,4A,4A,4A)$	2	40
	$(2B,2B,2B,2A,2A)$	1	40	$(2B,2B,2B,2A,4A)$	1	136
	$(2B,2B,2B,4A,4A)$	1	360			

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