Analysis of the Generalized Certain Nested Repeated Measures Models

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Abstract: Previously a general method to analyze a nested repeated measure model was developed when the covariance matrix had a certain pattern. A case of being a number of sub-individuals of a particular individual, such as sub-field or other types of offsprings, receives several treatments. As a consequence, the observations are correlated with certain covariance matrix pattern and such a model is known as nested repeated measures model (NRMM). In this paper, a weaker assumption is used when the covariance matrix is arbitrary and has no specific pattern. Independent normally distributed individuals are taken with their own mean and common positive definite covariance matrix. It is aimed to test hypotheses about the mean. Two techniques are used for testing. The first is based on the multivariate one sample model (MOSM), when each individual receives the same treatments and hence has the same mean vector, whilst the second is based on the multivariate linear model (MLM). Different individuals receive different treatments and hence have different mean vectors. For each technique a uniformly most powerful (UMP) invariant size α test is found.

Keywords: Multivariate Linear Model, Multivariate One Sample Model, Nested Repeated Measures Model

1. Introduction

This work is an extension of work on the generalized repeated measures model of Gabbara (2005). Here, we consider a design of experiments that occurs in the analysis of variance (ANOVA) when a particular individual (person, rat, field, etc.) has a number of sub-individuals (children, offspring, subfield, etc.) and each sub-individual receives several treatments (or has several measurements, e.g., height, weight, yield, etc.) and hence the observations cannot be assumed independent as they are assumed in the usual independent measures designs (Overall, 1996; Diana, 2015; Dien, 2016). Also, a new toolbox has been conducted for dependent repeated measures that neuroimaging data was its one application (McFarquhar, McKie, & Emsley, 2016). There is modification heteroscedastic one-way and notes on the modification two-way multivariate analysis of variance (MANOVA) and it has shown how it can be improved (Zhang & Xiao, 2012; Zhang & Liu, 2013) as well as proposing a parametric bootstrap test in heteroscedastic two-way MANOVA (Xu, 2015).

We assume throughout this work that each individual has the same number d, of sub-individuals and each sub-individual receives the same number r, of treatments. Let Y_{ijk} be the k^{th} observation on the j^{th} sub-individual from the i^{th} individual ($1 \le i \le n, 1 \le j \le d, 1 \le k \le r$), and let $Y_i = \left(Y_{i11}, \cdots, Y_{idr}\right)'$ be the vector of observations on the i^{th} individual $i = 1,2,\cdots,n$. It is assumed that the Y_i's are independently normally distributed with mean μ_i and common arbitrary covariance matrix $\Sigma > 0$. Arnold (1979) and Gabbara (1985) developed a general method to analyze a nested repeated measures model (NRMM) when the covariance matrix $\sum$ has the pattern

$$\Sigma = \Sigma(\sigma^2, \rho_1, \rho_2) = \sigma^2 \begin{bmatrix} A_2(\rho_2) & A_1(\rho_1) & \cdots & A_1(\rho_1) \\ A_1(\rho_1) & A_2(\rho_2) & \cdots & A_1(\rho_1) \\ \vdots & \vdots & \ddots & \vdots \\ A_1(\rho_1) & A_1(\rho_1) & \cdots & A_2(\rho_2) \end{bmatrix}$$

$$= \sigma^2 \{[A_2(\rho_2) - A_1(\rho_1)] \otimes I_d + A_1(\rho_1) \otimes J_d\} \quad [1]$$

where

$$A_2(\rho_2) = \begin{bmatrix} 1 & \rho_2 & \cdots & \rho_2 \\ \rho_2 & 1 & \cdots & \rho_2 \\ \vdots & \vdots & \ddots & \vdots \\ \rho_2 & \rho_2 & \cdots & 1 \end{bmatrix} = (1-\rho_2) I_r + \rho_2 J_r,\ A_1(\rho_1) = \begin{bmatrix} \rho_1 & \cdots & \rho_1 \\ \vdots & \ddots & \vdots \\ \rho_1 & \cdots & \rho_1 \end{bmatrix} \quad [2]$$

and

$$-\frac{1}{r-1} < \rho_2 < 1,\ -\frac{1}{r(d-1)}[1+(r-1)\rho_2] < \rho_1 < \frac{1}{r}[1+(r-1)\rho_2] \quad [3]$$

To find optimal (i.e., UMP invariant, UMP unbiased, etc.) procedures for testing a large class of hypotheses about the μ_i. For the main body of this paper, we replace the assumption in (1) with the weaker assumption that $\sum > 0$ is arbitrary (i.e., un-pattern). We are primarily concerned in testing hypotheses about μ_i. Two techniques are used to do the testing. The first is based on the multivariate one sample model (MOSM), for testing that $A\mu = b$, where A is an $s \times dr$ matrix of rank s, when each individual receives the same treatments and hence has the same mean vector, i.e., $\mu_i = \mu$ for each i (Akritas & Arnold, 1994; Anderson, 1984; Morrison, 1990). The second is based on testing the generalized linear hypothesis that $\mu A \in W$, where A is a known matrix and W is a subspace of V in the multivariate linear model (MLM). This is used when different individuals receive different treatments and hence have different mean vectors. For each of the two techniques, we find the UMP invariant size α test (Akritas & Arnold, 1994; Anderson, 1984; Morrison, 1990). We call the model considered in this work the generalized nested repeated measures model (GNRMM).

2. Notations

In this section we define the notations used in this paper. Let V be a subspace of R^m, and let $Y \in R^m$. Then $P_V Y$ is the projection of Y on V, and $V^{\perp}$ is the orthogonal complement of V. If

W is a subspace of V, then $V \mid W = V \cap W^{\perp}$.

The following special matrices are used. 1_n is the $n \times 1$ vector of $1's$, I_n is the $n \times n$ identify matrix, $J_n = 1_n 1_n'$ is the $n \times n$ matrix of $1's$,

$$M_n = (1/n) 1_n 1_n' = (1/n) J_n \,,\ N_n = I_n - M_n \qquad [4]$$

If A and $B = (b_{ij})$ are $m \times p$ and $n \times q$ matrices, then the Kronecker product of A and B, written as $A \otimes B$ is the $mn \times pq$ matrix $C = (C_{ij})$, where $C_{ij} = b_{ij} A$, $i = 1, \cdots, n;\ j = 1, \cdots, q$, (Dauxois, Romain & Viguier-Pla, 1994; Rogers (1984). A' is the transpose of A and A^{-1} is the inverse of A when A is a nonsingular square matrix.

3. Testing Hypotheses About μ

We now consider testing two different hypotheses about μ for the GNRMM.

3.1 Testing That $A\mu = b$.

The first testing problem we consider is testing that $A\mu = b$, where A is an $s \times dr$ matrix of rank s. For this problem the MOSM technique is used in which we observe $Y_1, \cdots, Y_n$ independently distributed $dr-$dimensional random vectors such that

$$Y_i \sim N_{dr}(\mu, \Sigma), \quad -\infty < \mu < \infty, \ \Sigma > 0 \qquad [5]$$

The test for this problem is UMP invariant size α test for testing that $A\mu = b$ against μ unrestricted in the MOSM and is given the following from

$$F_1 = \frac{(n-s)}{s(n-1)} (A\overline{Y} - b)'(ASA')^{-1}(A\overline{Y} - b)$$

$$\Phi_1(F_1) = \begin{cases} 1 & if \quad F_1 > F^{\alpha}_{s,n-s} \\ 0 & if \quad F_1 \le F^{\alpha}_{s,n-s} \end{cases} \qquad [6]$$

where Y is $n \times dr$ matrix whose rows are Y_i and

$$\overline{Y}' = \frac{1}{n} 1_n' Y, \ (n-1)S = Y'Y - n(\frac{1}{n}1_n' Y)'(\frac{1}{n}1_n' Y) = Y'(I_n - M_n)Y = Y'N_n Y \qquad [7]$$

This test is used when each sub-individual receives the same treatments and hence has the same mean vector. Applications of this technique are shown below.

3.1.1 The 1- way ANOVA Model

Consider a 1-way ANOVA model with r treatment levels in which each of d sub-individuals in each of n individuals receive each treatment level. Let $Y_i = (Y_{i11}, \cdots, Y_{idr})'$ be the vector of observations on the i^{th} individual $i = 1, \cdots, n$. Then the model would be given by

$$Y_{ijk} = \theta + \alpha_k + e_{ijk}, \ \sum_{1}^{b} \alpha_k = 0, \ e_i = (e_{i11}, \cdots, e_{idr})' \sim N_{dr}(0, \Sigma), \ \Sigma > 0 \qquad [8]$$

and e_i are independent. (Note that, we have replaced the assumption of application 1 of the NRMM of Gabbara (1985), i.e. $e_i \sim N_{dr}(0, \Sigma)$, where Σ has the pattern given in (1), with a weaker assumption that Σ is un-pattern.). We want to test that $\alpha_k = 0$. Let

$$\mu = E(Y_i) = \begin{pmatrix} \theta + \alpha_1 \\ \theta + \alpha_2 \\ \vdots \\ \theta + \alpha_r \end{pmatrix} = (\theta 1_r + \alpha) \otimes 1_d, \quad i = 1, \cdots, n \qquad [9]$$

where $\alpha = (\alpha_1, \cdots, \alpha_r)' \perp 1_r$. Then $Y_i \sim N_{dr}(\mu, \Sigma)$, independent. Let A be an $(r-1) \times dr$ matrix such that

$$A = (1/d)(I_{r-1} : -1_{r-1}) \otimes 1'_d \qquad [10]$$

Then

$$A\mu = [(1/d)(I_{r-1} : -1_{r-1}) \otimes 1'_d]\mu = (\alpha_1 - \alpha_r, \alpha_2 - \alpha_r, \cdots, \alpha_{r-1} - \alpha_r)'$$

Therefore, the $\alpha_k = 0$ if and only if $A\mu = 0$. Hence, according to (6), the UMP invariant size α test that the $\alpha_k = 0$ for the 1-way ANOVA model with un-pattern covariance matrix $\Sigma > 0$ and $n = n$, $s = r - 1$ and A as defined in (10) is

$$F_1 = \frac{n - (r-1)}{(r-1)(n-1)} (A\bar{Y})'(ASA')^{-1}(A\bar{Y})$$

$$\Phi_1(F_1) = \begin{cases} 1 & if \quad F_1 > F^{\alpha}_{r-1, n-(r-1)} \\ 0 & if \quad F_1 \leq F^{\alpha}_{r-1, n-(r-1)} \end{cases} \qquad [11]$$

We note that this test is different from the test that was derived for the 1-way ANOVA model (application 1) of the NRMM of Gabbara (1985). We also note that we need to assume that $n > dr > 1$ to do the test in this work. Whereas, it is only necessary to assume that $n > 1$ to do the test suggested in Gabbara (1985).

3.1.2 The 2-way ANOVA Model with Interaction

As a second application of the first technique, we consider a 2-way ANOVA model with interaction with b rows and c columns in which each of d sub-individuals in each of n individuals receives every pair of treatment levels. Let $Y_i = (Y_{i111}, \cdots, Y_{idrc})'$ be the vector of observations on the i^{th} individual, $i = 1, \cdots, n$. Then the model is given by

$$Y_{ijhl} = \theta + \alpha_h + \beta_l + \gamma_{hl} + e_{ijhl}, \quad [12]$$

$$\sum_h \alpha_h = 0,\ \sum_l \beta_l = 0,\ \sum_h \gamma_{hl} = 0 = \sum_l \gamma_{hl},\ e_i = (e_{i111}, \cdots, e_{idrc})' \sim N_{drc}(0, \Sigma)$$

and the e_i are independent. Let

$$\mu_i = E(Y_i) = \begin{bmatrix} (Y_{i111}, \cdots, Y_{i11c})' \\ \vdots \\ (Y_{i1r1}, \cdots, Y_{i1rc})' \\ \vdots \\ (Y_{id11}, \cdots, Y_{id1c})' \\ \vdots \\ (Y_{idr1}, \cdots, Y_{idrc})' \end{bmatrix} = \begin{bmatrix} (\theta + \alpha_1 + \beta_1 + \gamma_{11}, \cdots, \theta + \alpha_1 + \beta_c + \gamma_{1c})' \\ \vdots \\ (\theta + \alpha_r + \beta_1 + \gamma_{r1}, \cdots, \theta + \alpha_r + \beta_c + \gamma_{rc})' \\ \vdots \\ (\theta + \alpha_1 + \beta_1 + \gamma_{11}, \cdots, \theta + \alpha_1 + \beta_c + \gamma_{1c})' \\ \vdots \\ (\theta + \alpha_r + \beta_1 + \gamma_{r1}, \cdots, \theta + \alpha_r + \beta_c + \gamma_{rc})' \end{bmatrix}$$

$$= [\theta 1_{rc} + 1_c \otimes \alpha + \beta \otimes 1_r + \gamma] \otimes 1_d$$

$$= \theta 1_c \otimes 1_r \otimes 1_d + 1_c \otimes \alpha \otimes 1_d + \beta \otimes 1_r \otimes 1_d + \gamma \otimes 1_d \quad [13]$$

where $\alpha = (\alpha_1, \cdots, \alpha_r)' \perp 1_r$, $\beta = (\beta_1, \cdots, \beta_c)' \perp 1_c$, $\gamma = (\gamma_{11}, \cdots, \gamma_{1c}, \cdots, \gamma_{rc})' \perp$ to every column of the matrix $1_c \otimes 1_r$ and to every column of the matrix $1_c \otimes 1_r$.

(i) Testing that $\alpha_h = 0$. Let A_1 be $(r-1) \times drc$ such that

$$A_1 = (1/dc)1'_c \otimes [I_{r-1} : -1_{r-1}] \otimes 1'_d \quad [14]$$

And using the orthogonally of β with 1_r and the orthogonally of γ with every column of the matrix $1_c \otimes 1_r$, then

$$A_1 \mu = \left((1/dc)1'_c \otimes [I_{r-1} : -1_{r-1}] \otimes 1'_d\right)\mu$$
$$= \left((1/dc)1'_c \otimes [I_{r-1} : -1_{r-1]} \otimes 1'_d\right)\left(\theta 1_c \otimes 1_r \otimes 1_d + 1_c \otimes \alpha \otimes 1_d + \beta \otimes 1_r \otimes 1_d + \gamma \otimes 1_d\right)$$
$$= (\alpha_1 - \alpha_r, \alpha_r - \alpha_r, \cdots, \alpha_{r-1} - \alpha_r)'$$

For illustration, take $d = 2$, $r = 3$ and $c = 4$ then (13) becomes

$$\mu = \theta 1_4 \otimes 1_3 \otimes 1_2 + 1_4 \otimes \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \otimes 1_2 + \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} \otimes 1_3 \otimes 1_2 + \begin{pmatrix} \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{14} \\ \gamma_{21} \\ \\ \gamma_{34} \end{pmatrix} \otimes 1_2 \qquad [15]$$

Using (14) and (15), then

$$A_1\mu = \left((1/8)1_4' \otimes [I_2 : -1_2] \otimes 1_2'\right) \left(\mu = \theta 1_4 \otimes 1_3 \otimes 1_2 + 1_4 \otimes \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \otimes 1_2 + \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} \otimes 1_3 \otimes 1_2 + \begin{pmatrix} \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{14} \\ \gamma_{21} \\ \\ \gamma_{34} \end{pmatrix} \otimes 1_2 \right)$$

$$= \theta[I_2 : -1_2] \otimes 1_3 + [I_2 : -1_2] \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} + \left((1/4)1_4' \otimes [I_2 : -1_2]\right) \begin{pmatrix} \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{14} \\ \gamma_{21} \\ \\ \gamma_{34} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1 - \alpha_3 + (1/4)(\gamma_{11} + \gamma_{12} + \gamma_{13} + \gamma_{14}) - (1/4)(\gamma_{31} + \gamma_{32} + \gamma_{33} + \gamma_{34}) \\ \alpha_2 - \alpha_3 + (1/4)(\gamma_{21} + \gamma_{22} + \gamma_{23} + \gamma_{24}) - (1/4)(\gamma_{31} + \gamma_{32} + \gamma_{33} + \gamma_{34}) \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1 - \alpha_3 \\ \alpha_2 - \alpha_3 \end{pmatrix}$$

Therefore, the $\alpha_h = 0$ if and only if $A\mu = 0$. Hence, according to (6), the UMP invariant size α test that the $\alpha_h = 0$ for the 2-way ANOVA model with un-pattern covariance matrix $\Sigma > 0$ and $n = n,\ s = r - 1$ and A_1 as defined in (10) is

$$F_1 = \frac{n - (r-1)}{(r-1)(n-1)} (A_1\bar{Y})'(A_1 S A_1')^{-1}(A_1\bar{Y})$$

$$\Phi(F_1)=\begin{cases}1 & if \quad F_1 > F^{\alpha}_{r-1,n-(r-1)}\\ 0 & if \quad F_1 \le F^{\alpha}_{r-1,n-(r-1)}\end{cases} \qquad [16]$$

(ii) Testing that $\beta_1 = 0.$ In a similar way to **(i)** above with

$A_2 = (1/dr)[I_{c-1} : -1_{c-1}] \otimes 1'_r \otimes 1'_d$

the UMP invariant size α test that the $\beta_l = 0$ is

$$F_2 = \frac{n-(c-1)}{(c-1)(n-1)}(A_2\bar{Y})'(A_2SA_2')^{-1}(A_2\bar{Y})$$

$$\Phi(F_2)=\begin{cases}1 & if \quad F_2 > F^{\alpha}_{c-1,n-(c-1)}\\ 0 & if \quad F_2 \le F^{\alpha}_{c-1,n-(c-1)}\end{cases} \qquad [17]$$

(iii) Testing that $\gamma_{hl} = 0.$ We can do this test in two different ways

(a) Let A_{31} be $(r-1)c \times drc$ matrix such that

$$A_{31} = (1/d)\, I_c \otimes [I_{r-1} : -1_{r-1}] \otimes 1'_d \qquad [18]$$

when $d = 2,\ r = 3$ and $c = 4$, then

$$A_{31} = (1/2)\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \otimes \begin{pmatrix}1&0&-1\\0&1&-1\end{pmatrix} \otimes (1 \quad 1)$$

$$= (1/2)\begin{pmatrix}1&0&0&0&0&0&0&0&-1&0&0&0\\0&1&0&0&0&0&0&0&0&-1&0&0\\0&0&1&0&0&0&0&0&0&0&-1&0\\0&0&0&1&0&0&0&0&0&0&0&-1\\0&0&0&0&1&0&0&0&-1&0&0&0\\0&0&0&0&0&1&0&0&0&-1&0&0\\0&0&0&0&0&0&1&0&0&0&-1&0\\0&0&0&0&0&0&0&1&0&0&0&-1\end{pmatrix} \otimes (1 \quad 1) \qquad [19]$$

Hence, using (15) and (19), then

$$A_{31}\mu = \left(\gamma_{11}-\gamma_{31}, \gamma_{12}-\gamma_{32}, \gamma_{13}-\gamma_{33}, \cdots, \gamma_{24}-\gamma_{34}\right)'$$

Therefore, the $\gamma_{hl} = 0$ if and only if $A_{31}\mu = 0$. Hence according to (6), the UMP invariant size α test that the $\gamma_{hl} = 0$ for the 2-way ANOVA model with un-pattern covariance matrix $\Sigma > 0$ and $n = n,\ s = c(r-1)$ and A_{31} as defined in (18)

is

$$F_{31} = \frac{n - c(r-1)}{c(r-1)(n-1)} (A_{31}\bar{Y})'(A_{31}SA'_{31})^{-1}(A_{31}\bar{Y})$$

$$\Phi(F_{31}) = \begin{cases} 1 & if \quad F_{31} > F^{\alpha}_{r(c-1),n-r(c-1)} \\ 0 & if \quad F_{31} \le F^{\alpha}_{r(c-1),n-r(c-1)} \end{cases} \qquad [20]$$

(b) Let A_{32} be $r(c-1) \times drc$ matrix such that

$$A_{32} = (1/d)[I_{c-1} : -1_{c-1}] \otimes I_r \otimes 1'_d \qquad [21]$$

when $d = 2,\ r = 3$ and $c = 4$, then

$$A_{32} = (1/2)\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$= (1/2)\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \end{pmatrix} \qquad [22]$$

Hence, using (15) and (21), then

$$A_{32}\mu = \left(\gamma_{11}-\gamma_{14}, \gamma_{12}-\gamma_{14}, \cdots, \gamma_{31}-\gamma_{34}\right)'$$

Therefore, the $\gamma_{hl} = 0$ if and only if $A_{32}\mu = 0$. Hence according to (6), the UMP invariant size α

test that the $\gamma_{hl} = 0$ for the 2-way ANOVA model with un-pattern covariance matrix $\Sigma > 0$ and $n = n, s = r(c-1)$ and A_{32} as defined in (21)

is

$$F_{32} = \frac{n - r(c-1)}{r(c-1)(n-1)}(A_{32}\bar{Y})'(A_{32}SA'_{32})^{-1}(A_{32}\bar{Y})$$

$$\Phi(F_{32}) = \begin{cases} 1 & if \quad F_{32} > F^{\alpha}_{r(c-1),n-r(c-1)} \\ 0 & if \quad F_{32} \le F^{\alpha}_{r(c-1),n-r(c-1)} \end{cases} \qquad [23]$$

3.2 Testing That $\mu A \in W \subset V$

In this section, we use the MLM technique to make hypothesis testing about μ. Let μ be an $n \times dr$ matrix, and let V be a p - dimensional subspace of R^n. We say that $\mu \in V$ if the columns of μ are in V. Let Y be an $n \times dr$ random matrix such that

$$Y = \begin{bmatrix} Y'_1 \\ \vdots \\ Y'_n \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu'_1 \\ \vdots \\ \mu'_n \end{bmatrix}, \; Y'_i = (Y_{i11}, \cdots, Y_{idr}), \quad \mu'_i = (\mu_{i11}, \cdots, \mu_{idr}) \qquad [24]$$

The (MLM) is the model in which we observe

$$Y \sim N_{n,dr}(\mu, I, \Sigma), \; \mu \in V, \; \Sigma > 0$$

Then, Y_i are independent and $Y_i \sim N_{dr}(\mu_i, \Sigma)$. In the above formulation of the linear model, therefore, the independent replication by the rows of the Y matrix.

We now consider the problem of testing the null hypothesis that $\mu \in W$ against the alternative that $\mu \in V$, where W is a k-dimensional subspace of V. We assume that $n - p \ge dr$, so that $\hat{\Sigma}$ is positive definite. Let

$$S_2 = Y'P_{V|W}Y, \; S_3 = Y'P_{V^{\perp}}Y = (n-p)\hat{\Sigma}, \; \delta = \mu'P_{V|W}\mu \qquad [25]$$

(Note that S_2, S_3 and δ are all $dr \times dr$ matrices). Let $b = \min\{p-k, dr\}$. A maximal invariant for the testing problem $\mu \in W$ against $\mu \in V$ is the set of nonzero roots $t_1 \ge \cdots \ge t_b$ of

$$|A'S_2A - tA'S_3A| = 0 \qquad [26]$$

Case I: When $b = 1$, i.e., the case in which there is only one nonzero root t_1.

In this case

$$t_1 = \frac{\|P_{V|W} Y\|^2}{\|P_{V^{\perp}}\|^2}, \qquad \frac{n-p}{p-k} t_1 \sim F_{p-k,n-p} \tag{27}$$

where $p-k = \dim(V \mid W)$, $n-p = \dim(V^{\perp})$ and the UMP invariant size α test is

$$\Phi(t_1) = \begin{cases} 1 & if \quad t_1 > \dfrac{p-k}{n-p} F_{p-k,n-p}^{\alpha} \\ 0 & if \quad t_1 < \dfrac{p-k}{n-p} F_{p-k,n-p}^{\alpha} \end{cases} \tag{28}$$

Case II: When $b > 1$, there is no UMP invariant test. However, the following four tests are available:

$$\lambda_1 = \prod_{i=1}^{b} (1+t_i) = \frac{|A'S_2A + A'S_3A|}{|A'S_3A|}, \qquad \lambda_2 = \sum_{i=1}^{b} t_i = tr(A'S_2A)(A'S_3A)^{-1},$$

$$\lambda_3 = t_1, \quad \lambda_4 = \sum_{i=1}^{b} \frac{t_i}{1+t_i} = tr(A'S_2A)(A'S_2A + A'S_3A)^{-1}$$

$$\Phi_i(\lambda_i) = \begin{cases} 1 & if \quad \lambda_i > c_i^{\alpha} \\ 0 & if \quad \lambda_i < c_i^{\alpha} \end{cases} \quad \text{for } i = 1,2,3,4 \tag{29}$$

where c_i^{α} is the upper α point of distribution of λ_i,

$$c_1^{\alpha} \approx \exp\left(\frac{\chi^{2\ \alpha}_{r(p-k)}}{n-p-\dfrac{r-(p-k)+1}{2}}\right), \quad c_{2,4}^{\alpha} \approx \frac{\chi^{2\ \alpha}_{r(p-k)}}{n-p} \quad and \quad c_3^{\alpha} \approx \frac{u^{\alpha}_{r,(p-k)}}{n-p}$$

All the four tests are invariant, unbiased (Bradley & Russell, 1998; Bradley & Orfaly, 1999; Das Gupta, Anderson, & Mudholkar (1964) and admissible (Schwartz, 1967). These tests are used when different individuals receive different treatments and hence have different mean vectors. Application of this technique is shown below.

3.2.1 The 2-way ANOVA Model with Interaction (Another Version)

Consider now a 2-way ANOVA model with r rows and c columns in which each of n individuals receives only one row treatment and each of d sub-individuals receives all column treatments (this could occur, for example, if rows represent race, sex, degrees of illness, etc.). Let

$Y_{ih}=(Y_{ih11},\cdots,Y_{ihdc})'$ be the vector of observations on the i^{th} person who receives the h^{th} row treatment, $h=1,\cdots,r,\ i=1,\cdots,n$. The model would then be

$$Y_{ihjl}=\theta+\alpha_h+\beta_l+\gamma_{hl}+e_{ihjl}$$

$$\sum_h\alpha_h=0,\ \sum_l\beta_l=0,\ \sum_h\gamma_{hl}=0=\sum_l\gamma_{hl},\ e_{ih}=(e_{ih11},\cdots,e_{ihdc})'\sim N_{drc}(0,\Sigma) \quad [30]$$

and e_{ih} are independent. Now, let

$$\alpha=\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_r\end{bmatrix},\ \beta=\begin{bmatrix}\beta_1\\ \vdots\\ \beta_c\end{bmatrix},\ \gamma_h=\begin{bmatrix}\gamma_{h1}\\ \vdots\\ \gamma_{hc}\end{bmatrix},\ \gamma=\begin{bmatrix}\gamma_1\\ \vdots\\ \gamma_r\end{bmatrix} \quad [31]$$

Then

$$\mu_{ih}=E(Y_{ih})=[(\theta+\alpha_h)1_c+\beta+\gamma_h]\otimes 1_d \quad [32]$$

Let

$$Y=\begin{bmatrix}Y'_{11}\\ \vdots\\ Y'_{1r}\\ Y'_{21}\\ \vdots\\ Y'_{nr}\end{bmatrix},\qquad \mu=E(Y)=\begin{bmatrix}\mu'_{11}\\ \vdots\\ \mu'_{1r}\\ \mu'_{21}\\ \vdots\\ \mu'_{nr}\end{bmatrix}$$

Using (32), then

$$\mu=\begin{bmatrix}\mu_{i1}\\ \vdots\\ \mu_{ir}\end{bmatrix}=\begin{bmatrix}[(\theta+\alpha_1)1'_c+\beta'+\gamma'_1]\otimes 1'_d\\ \vdots\\ [(\theta+\alpha_r)1'_c+\beta'+\gamma'_h]\otimes 1'_d\end{bmatrix} \quad [33]$$

Then $Y\sim N_{nr,dc}(\mu,I,\Sigma)$. Let V be the $r-$dimensional subspace of R^{nr} consisting of vectors whose first n components are the same, whose next n components are the same, etc. Then the only restriction on μ is that $\mu\in V$ so that this model is (MLM).

(i) Testing that $\alpha_h=0$. We first consider testing the hypothesis that $\alpha_h=0$. Using (32) then

$$\mu'_i 1_{dc}=dc(\theta+\alpha_1,\cdots,\theta+\alpha_r)'=dc(\theta 1_r+\alpha)$$

where $\alpha = (\alpha_1, \cdots, \alpha_r)' \perp 1_r$. Now, let W_1 be the $1-$dimensional subspace of V consisting of vectors all of whose elements are the same. Then

$$\alpha_h = 0 \quad iff \quad \mu_i' 1_{dc} \in W_1$$

Therefore, this hypothesis is in the form of the generalized multivariate linear hypothesis. Also, $s = 1$ for this model so that there is a UMP invariant size α test. Let

$$S_2 = Y' P_{V|W_1} Y = \sum_{h=1}^{r} n(\bar{Y}_{h..} - \bar{Y}_{...})(\bar{Y}_{h..} - \bar{Y}_{...})$$

$$S_3 = Y' P_{V^{\perp}} Y = \sum_{i=1}^{n} \sum_{h=1}^{r} (\bar{Y}_{ih.} - \bar{Y}_{i..})(\bar{Y}_{ih.} - \bar{Y}_{i..}) \qquad [34]$$

Then, according to (26) a maximal invariant for this model is

$$t = \frac{1'_{dc} S_2 1_{dc}}{1'_{dc} S_3 1_{dc}} \qquad [35]$$

Since $\dim(V \mid W) = r - 1$ and $\dim(V^{\perp}) = rn - r = r(n-1)$ the UMP invariant size α test is

$$\Phi(t_1) = \begin{cases} 1 & if \quad t_1 > \dfrac{r-1}{r(n-1)} F^{\alpha}_{r-1, r(n-1)} \\ 0 & if \quad t_1 \le \dfrac{r-1}{r(n-1)} F^{\alpha}_{r-1, r(n-1)} \end{cases} \qquad [36]$$

(ii) Testing that $\beta_1 = 0$. Now, consider testing that $\beta_1 = 0$. Let X be a basis matrix for the space orthogonal to 1_c. Then

$$\mu X = \begin{bmatrix} \beta_1 + \gamma_{11}, \cdots, \beta_c + \gamma_{1c} \\ \beta_1 + \gamma_{21}, \cdots, \beta_c + \gamma_{2c} \\ \\ \beta_1 + \gamma_{r1}, \cdots, \beta_c + \gamma_{rc} \end{bmatrix} X$$

Let W_2 be the $(r-1)-$dimensional subspace consisting of vectors whose components sum to 0. Then

$\beta_1 = 0$ if and only if $\mu X \in W_2$

so that this hypothesis is also in the form of the generalized multivariate linear hypothesis. In this case, $p - k = 1$, so that there is again a UMP size α F-test for this hypothesis. A maximal invariant for this problem is the one nonzero root of

$$\left| X'Y'P_{V|W_2} YX - tX'S_3X \right| = 0 \qquad [37]$$

which is

$$t = 1'_{rn} YX (X'S_3X)^{-1} X'Y' 1_{rn} / rn \qquad [38]$$

To show that t of (38) is the nonzero root of (37). Let Z be the orthogonal basis matrix for $V \mid W_2$. Then

$$Z = (1/\sqrt{rn}) 1_{rn}, \quad P_{V|W_1} = ZZ' \qquad [40]$$

Then

$$0 = \left| X'Y'P_{V|W_2} YX - tX'S_3X \right| = \left| X'Y'ZZ'YX - tX'S_3X \right| = \left| H'H - tX'S_3X \right|$$

$$= \left| H'(X'S_3X)^{-1} H - tI \right|$$

where $H = Z'YX$. Hence, the nonzero root of (40) is

$$t = H(X'S_3X)^{-1} H'$$

$$= Z'YX (X'S_3X)^{-1} X'Y'Z$$

$$= 1'_{rn} YX (X'S_3X)^{-1} X'Y' 1_{rn} / rn$$

Therefore, the UMP invariant size α test for this model is

$$\Phi(t_1) = \begin{cases} 1 & \text{if} \quad t_1 > \dfrac{c-1}{r(n-1)-c+2} F^{\alpha}_{c-1, r(n-1)-c+2} \\ 0 & \text{if} \quad t_1 \le \dfrac{c-1}{r(n-1)-c+2} F^{\alpha}_{c-1, r(n-1)-c+2} \end{cases} \qquad [42]$$

(iii) Testing that $\gamma_{ij} = 0$. Finally, we consider testing that $\gamma_{ij} = 0$. Let W_1 be defined as in (i). Then

$$\gamma_{hl} = 0 \quad iff \quad \mu X \in W_1$$

Therefore, this hypothesis is also a generalized multivariate linear hypothesis. For this problem $b = \min\{r-1, c-1\} > 1$, so that there is no UMP invariant size α test. A maximal invariant is the set of nonzero roots of

$\left| X'S_2X - tX'S_3X \right| = 0$ where S_2 and S_3 are defined in (34). And any one of the four tests Φ_i, $i = 1,2,3,4$ of (29) can be used this hypothesis and

$$\Phi(\lambda_i)=\begin{cases}1 & if \quad \lambda_i > c_i^{\alpha}\\ 0 & if \quad \lambda_i < c_i^{\alpha}\end{cases}$$

Where,

$$c_1^{\alpha}\approx \exp\left(\frac{\chi^{2\ \alpha}_{(c-1)(r-1)}}{nr-\frac{r+c+1}{2}}\right),\quad c_{2,4}^{\alpha}\approx\frac{\chi^{2\ \alpha}_{(c-1)(r-1)}}{r(n-1)}\quad and \quad c_3^{\alpha}\approx\frac{u^{\alpha}_{(c-1)(r-1)}}{r(n-1)}$$

It appears that the maximal invariants for the last two testing problems of this section could depend on the choice of the basis matrix, in fact they do not. In particular, it is not necessary to choose an orthonormal basis matrix as it is shown below.

If X and X^* are basis matrices for a subspace V, then there exists an invertible matrix B such that

$$X^*=XB$$

The maximal invariant for testing $\beta_1=0$ in the (GNRMM) is given in (42) and X is the basis matrix for the space orthogonal to 1_c. Suppose, we use X^* as a basis matrix for the space orthogonal to 1_c. Then the invariant is

$$t^*=1'_{rn}YX^*(X^{*'}S_3X^*)^{-1}X^{*'}Y'1_{rn}\big/rn$$

Then using (37) in (38), we get

$$\begin{aligned}t^*&=1'_{rn}YXB(B'X'S_3XB)^{-1}B'X'Y'1_{rn}\big/rn\\ &=1'_{rn}YXBB^{-1}(X'S_3X)^{-1}B'^{-1}B'X'Y'1_{rn}\big/rn\\ &=1'_{rn}YX(X'S_3X)^{-1}X'Y'1_{rn}\big/rn\\ &=t\end{aligned}$$

Now, for testing that $\gamma_{hl}=0$, the maximal invariant is the set of nonzero roots of

$$\left|X'S_2X-tX'S_3X\right|=0$$

When we use X^*, we get

$$\left|X^{*'}S_2X^*-t^*X^{*'}S_3X^*\right|=0 \quad\Leftrightarrow\quad \left|B'X'S_2XB-t^*B'X'S_3XB\right|=0$$

$$\left|B'\right|\left|X'S_2X-t^*X'S_3X\right|\left|B\right|=0 \quad\Leftrightarrow\quad \left|X'S_2X-t^*X'S_3X\right|=0$$

and $B > 0$ and $|B| \neq 0$.

Hence, they have the same set of nonzero roots. That is, the tests do not depend on which basis is chosen.

4. Conclusion

1. We note that the test given in (11) is different from the test that was derived for the (NRMM) of Gabbara (1985), where for testing that the $\alpha_k = 0$, it was F statistics with $(r-1)$ and $(nd-1)(r-1)$; whereas in this work it is F statistics with $(r-1)$ and $(n-(r-1))$.
2. We note also that we need to assume that $n > dr$ to do the test in section 3.1.1 with weaker assumption, while it is only necessary to assume that $n > 1$ to do the test suggested in Gabbara (1985).
3. For testing that $\alpha_h = 0$ for the 2-way ANOVA model of section 3.2.1, we note that the drop in degrees of freedom from the OLM to the NRMM is much greater than it is for the GNRMM [$drc(n-1)$ to $r(c-1)$], but there is no drop at all from the NRMM to the GNRMM. That is, they are same as the test for the NRMM derived in Gabbara (1985).

4. For testing that $\beta_1 = 0$ for the 2-way ANOVA model of section 3.2.1, if we analyze the model by ordinary linear model (OLM) or as NRMM of Gabbara (1985) or as GNRMM of the present work, we note that the three F statistics are UMP invariant size α test, the numerator degrees of freedom for $drc(n-1)$, $(c-1)(dn-1)$ and $r(n-1)-c$, respectively. We note that the drop in degrees of freedom from OLM to NRMM is much smaller than the drop from NRMM to GRMM. This fact indicates that if we have independent data, the OLM is true and whenever we analyze the data using NRMM, we lose a moderate amount of power. However, if the covariance is patterned as (1), the NRMM is true, and we analyze the data using the GNRMM we lose considerably more power.

References

Akritas, M. G., & Arnold, S.F. (1994). Fully nonparametric hypothesis for factorial design i: Multivariate repeated measures designs. *JASA*, 89, 336-343.

Anderson, T.W. (2003). *An introduction to multivariate analysis*. 3rd Ed. John Wiley, New York.

Arnold, S. F. (1979). A coordinate-free approach to finding optimal procedures for repeated measures designs. *Ann. Statist*, 7 812–822.

Bradley, D. R., & Russell, R. L. (1998). Some cautions regarding statistical power in split-plot designs. Behavior Research Methods, Instruments, and Computers.

Bradley, D. R., & Orfaly, R. A. (1999). Statistical power of completely randomized and split-plot factorial designs. Behavioral Neuroscience.

Das Gupta, S., Anderson, T. W., & Mudholkar, G. S. (1964). Monotonicity of the power functions of some tests of the multivariate linear hypothesis. *Ann. Math, Stat*., 35, 200-205.

Dauxois, J., Romain, Y., & Viguier-Pla, S. (1994). Tensor products and statistics. *Linear Algebra and Its Applications*, 210:59-88.

Diana, G. (2015). Repeated measures analysis on determinant factors of enterprise value. *Procedia Economics and Finance*, 32, 338 – 344.

Dien, J. (2016). Best practices for repeated measures ANOVAs of ERP data: reference, regional channels, and robust ANOVAs, *International Journal of Psychophysiology*. doi: 10.1016/j.ijpsycho.2016.09.006.

Gabbara, S. D. (2005). Analysis of generalized repeated measures models. *Al-Mustansiria Journal of Science*, 16(4), 95-108.

Gabbara, S. D. (1985). Nested and Crossed repeated measures models, *Ph.D. Thesis*, The Pennsylvania State University, USA.

Konietschke, F., Bathke A. C. &, Harrar, S. W. (2015). Parametric and nonparametric bootstrap methods for general MANOVA. *Journal of Multivariate Analysis*, 140, 291–301.

McFarquhar, M., McKie, K., & Emsley, R. (2016). Multivariate and repeated measures (MRM): A new toolbox for dependent and multimodal group-level neuroimaging data, *NeuroImage*, doi: 1016/j.neuroimage.2016.02.053.

Morrison, D.F. (1990). *Multivariate Statistical Methods*. 3rd Edition. New York: McGraw-Hill.

Overall, J. E. (1996). How many repeated measurements are useful? *Journal of Clinical Psychology*, 52, 243-252.

Rogers, G. S. (1984). Kronecker products in ANOVA a first step, *The American Statistician*, 38 (3), 197-202.

Schwartz, R. (1967). Admissible tests in multivariate analysis. *Ann. Math. Stat*., 38,698-710.

Xu, L. W. (2015). Parametric bootstrap approaches for two-way MANOVA with unequal cell sizes and unequal cell covariance matrices. *Journal of Multivariate Analysis*, 133, 291–303.

Zhang, J.,T.& Xiao, S. (2012). A note on the modified two-way MANOVA tests, *Statistics and Probability Letters*, 82(3), 519–527. doi: 10.1016/j.spl.2011.12.005.

Zhang, J., T.& Liu, X.. (2013). A modified Bartlett test for heteroscedastic one-way MANOVA, *Metrika*, 76, 135–152. doi: 10.1007/s00184-011-0379-z.