A Short Review on *p*-Adic Numbers

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Abstract: In this review work we indicated that there are two different metrics to find the distance between any two rational numbers. One of these metrics is usual absolute value $|\cdot|_\infty$ and the other one is the p-adic absolute value $|\cdot|_p$, here p is a prime number. Most crucial property of this norm is that it satisfies the ultra-metric triangle inequality. In this work we gave some definitions and properties about both metric and ultra metric norms. Especially, we reviewed the construction of *p*-adic number field. Rational numbers have two types of completions; while one of them is real numbers field the other one is *p*-adic numbers field.
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1. Introduction

p-adic numbers were first described by German mathematician Kurt Hensel. About a century later, p-adic numbers was one of the most crucial topics of pure mathematics to investigate. After 1980s, many scientists recognized the importance of p-adic numbers and many researches studied on *p*-adic numbers and its various applications in their works (Arefeva et al., 1991; Rozikov 1998; Khrennikov, 1997, 2003 and 2004; Ganikhodjaev & Rozikov, 2009). Recently, p-adic numbers are keeping its importance to attract the mathemacians, and many other scientists with its applications in various areas. Also many books and PhD dissertations were published on p-adic numbers and p-adic analysis (e.g Koblitz, 1977; Robert, 2000; Katok, 2007; Rozikov, 2013; Dogan, 2015).

In the present review paper, to take the awareness of the mathematicians on *p*-adic numbers and its importance, I will use resources mentioned above and then I will give some fundamental notions, theorems and properties of the *p*-adic numbers. Furthermore, how to construct *p*-adic numbers field $\mathbf{Q}_p$ and some comparisons with real numbers is another issue the present paper deals with.

2. Some Definitions and Theorems

As it is known that real numbers $\mathbb{R}$ and *p*-adic numbers $\mathbb{Q}_p$ are obtained by the completion of rational numbers $\mathbb{Q}$. Each of these numbers determine the distance between a point on the number line and origin. Euclidean norm states the distance between origin and a point on the real number line with the absolute value at infinity, $\left(|\cdot|_\infty\right)$. In the usual absolute value $\left(|\cdot|_\infty\right)$, if we take any prime number p instead of infinity then we call this absolute value “p-adic norm” and denoted by

$|\cdot|_p$. Most useful and important of the p-adic metric is "it satisfies the strong triangle inequality." $|x+y|_p \leq \max\{|x|_p, |y|_p\}$, which is also called ultra-metric triangle inequality. This metric provided the second type of the completion of rational numbers that is called *p*-adic numbers field $\mathbb{Q}_p$.

Following proposition is very simple but it is very useful.

Proposition 2.1 (Katok 2007) If $\lim_{n\to\infty} x^n = 0$ then $\|x\| < 1$.

Proof: Let $\|x\| < 1$. $\lim_{n\to\infty} \|x^n\| = 0$ since $\|x^n\| = \|x\|^n$ i.e. $\lim_{n\to\infty} x^n = 0$. Let us assume that $\|x\| \geq 1$. Hence we get $\|x^n\| \geq 1$ for all positive n. So, it shows that $\lim_{n\to\infty} x^n \neq 0$. Therefore, this is a contradiction with $\lim_{n\to\infty} x^n = 0$. Then we conclude that the proposition holds.

Definition 2.2 If a norm holds the following inequality;

$$\|x+y\| \leq \max\{\|x\|, \|y\|\} \tag{0.1}$$

then this norm is called "non-Archimedean norm."

Following proposition gives us the condition of the non-Archimedean norm.

Proposition 2.3 (Katok 2007) Items below are identical

i) The norm $\|\cdot\|$ is non-Archimedean

ii) ii) $\|n\| \leq 1$ for all $n \in \mathbb{Z}$.

Proof: $(i) \Rightarrow (ii)$: We prove this assertion by induction method. Firstly, let

For $n = 1$, $\|1\| = 1 \leq 1$.

For $k = n-1$, $\|k\| \leq 1$.

From here let us show that $\|n\| \leq 1$. We can easily get that; $\|n\| = \|n-1+1\| \leq \max\{\|n-1\|, \|1\|\} = 1$ since $\|n-1\| = \|k\| \leq 1$ and $\|1\| = 1$. Therefore for all $n \in \mathbf{N}$ we get $\|n\| \leq 1$. For all $n \in \mathbf{Z}$, $\|n\| \leq 1$ since $\|-n\| = \|n\|$.

$(ii) \Rightarrow (i)$:

$$\|x+y\|^n = \|(x+y)^n\| = \left\|\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}\right\| \le \sum_{k=0}^{n}\left\|\binom{n}{k}\right\|\|x\|^k\|y\|^{n-k}$$
$$\le \sum_{k=0}^{n}\|x\|^k\|y\|^{n-k} \le (n+1)\left[\max\{\|x\|,\|y\|\}\right]^n \quad (0.2)$$

Then for all $n\in\mathbf{Z}$, we obtain that $\|x+y\|\le\sqrt[n]{n+1}\max\{\|x\|,\|y\|\}$. Hence, as $n\to\infty$, $\|x+y\|\le\max\{\|x\|,\|y\|\}$ since $\lim_{n\to\infty}\sqrt[n]{n+1}=1$.

This proposition provides us to get the difference between Archimedean and non-Archimedean norms. In this case for $x,y\in F, x\ne 0$, if there exists any positive integer n that satisfies the inequality $\|n\cdot x\|>\|y\|$, then this norm is called Archimedean norm. To indicate this for $x,y\in F$, let $\|y\|>\|x\|$, then we get $\|n\|\ge\frac{\|y\|}{\|x\|}>1$ for an existing positive integer n i.e. this norm is an Archimedean norm. Conversely, if a norm is an Archimedean then $\|n\|>1$ for a positive integer n. At the same time, as $k\to\infty$, $\|n\|^k\to\infty$ and for some k, we can get $\|n\|^k>\frac{\|y\|}{\|x\|}$ and it satisfies the Archimedean property $\|n^k x\|>\|y\|$ i.e.

$$\sup\{\|n\|: n\in\mathbf{Z}\}=+\infty \quad (0.3)$$

Proposition 2.4 (Katok 2007) Let F be a non-Archimedean field. For all $a,x\in F$, If the inequality $\|x-a\|<\|a\|$ holds then $\|x\|=\|a\|$.

Proof: From the strong triangle inequality, we get $\|x\|=\|x-a+a\|\le\max\{\|x-a\|,\|a\|\}=\|a\|$. On the other hand, it becomes $\|a\|=\|a-x+x\|\le\max\{\|x-a\|,\|x\|\}$. If $\|x-a\|>\|x\|$ then $\|a\|\le\|x-a\|$. This contradicts with the condition $\|x-a\|<\|a\|$. Therefore it becomes $\|x-a\|\le\|x\|$ and $\|a\|\le\|x\|$. Thus $\|x\|=\|a\|$.

Note 2.5 We can express the proposition above as follows:

Let $a,b\in F$ and $\|\cdot\|$ is a non-Archimedean norm on the field F. Therefore

$$\|a\|\ge\|b\|\Rightarrow\|a+b\|=\|a\|. \quad (0.4)$$

This property shows that in an ultra-metric space all triangles are isosceles and legs are longer than its base in length.

Proposition 2.5 (Katok 2007) Let the norm $(\|\cdot\|)$ be a non-Archimedean norm. Any point belonging to the open ball $B(a,r)=\{x:\|x-a\|<r\}$ is the center of this ball in field F, i.e. if $b\in B(a,r)$ then $B(b,r)=B(a,r)$. Same result also works for the closed ball in the field F.

Proof: Assume that $x\in B(b,r)$. From our assumption we get $|a-b|_p<r,\ |b-x|_p<r$ and using the strong triangle inequality, it is easy to get follows

$$|a-x|_p=|(a-b)+(b-x)|_p\le\max\left(|a-b|_p,|b-x|_p\right)<r.$$

Hence; $B(b,r)\subset B(a,r)$. By the same manner; easily it can be obtained that $b\in B(a,r)$, $a\in B(b,r)$ since $|a-b|_p<r$ for b. Then we get $B(a,r)\subset B(b,r)$ and we conclude that both balls are identical.

Next proposition proves that any norm is either Archimedean or non-Archimedean.

Proposition 2.6 (Katok 2007) Any two equivalent $\left(\|\cdot\|_1 \sim \|\cdot\|_2\right)$ norm in a field F, is either Archimedean or non-Archimedean.

Proof: If $\|\cdot\|_1 \sim \|\cdot\|_2$ then $\|x\|_1>1$ and $\|x\|_2>1$ for any integer x. To prove this, let us assume that $\|x\|_1>1$ and $\|x\|_2<1$. Therefore $\|x^n\|_1\to\infty$ and $\|x^n\|_2\to 0$ as $n\to\infty$. This shows that the sequence (x^n) is a Cauchy sequence due to $\|\cdot\|_1$ but not a Cauchy due to $\|\cdot\|_2$. This result contradicts with the equivalence of the norms. So, any norm is either Archimedean or non-Archimedean.

3. *p*-Adic Numbers Field ($\mathbf{Q}_p$) and Some Analysis on $\mathbf{Q}_p$

In this section we are going to review some important properties of the p-adic numbers field $\mathbf{Q}_p$. Usual absolute value $(|\cdot|)$ is a norm in the rational numbers field. The metric $d(x,y)=|x-y|$ determine the distance between an two points on the real number line which is called Euclidean metric. The completion of the rational numbers $\mathbf{Q}$ respect to this metric is real numbers field $\mathbf{R}$, i.e. $\bar{\mathbf{Q}}=\mathbf{R}$.

Naturally; we may inquire that is there any other metric different from usual metric and it should be completion of rational numbers. Answer of this question uncovered existence of another metric that different than the usual one. Following construction demonstrates that there exists a different way to measure the distance between any two rational number. Let p be a prime number. Hence;

$$x=p^{\mathrm{ord}_p(x)}\cdot\left(\sum_{k=0}^{\infty}x_k p^k\right) \tag{0.5}$$

and

$$ord_p x = \begin{cases} \text{The highest order of } p \text{ that } p \text{ divides } x,\ x \in \mathbb{Z}, \\ \text{If } x = \dfrac{a}{b},\ a,b \in \mathbb{Z}, b \neq 0 \text{ then } ord_p a - ord_p b. \end{cases} \quad (0.6)$$

From (0.5) and (0.6) we get;

$$|x|_p = \begin{cases} p^{-ord_p x} & \text{whenever } x \neq 0, \\ 0 & \text{whenever } x = 0. \end{cases} \quad (0.7)$$

We define the norm (0.7) as a transformation of $|\cdot|_p : \mathbf{Q} \to \mathbf{R}$. This norm is called *p-adic norm* in the rational numbers.

Remark 3.1 If $a,b \in \mathbf{N}$ then $a \equiv b (\text{mod } p^n) \Leftrightarrow |a-b|_p \leq \frac{1}{p^n}$ since when $a \equiv b (\text{mod } p^n)$, $a - b = p^n \cdot t$. From here, $|a-b|_p = |p^n \cdot t|_p \leq \frac{1}{p^n}$ is obtained.

Lemma 3.2 The *p*-adic $\left(|\cdot|_p\right)$ norm above is a non-Archimedean norm in rational numbers $\mathbf{Q}$, i.e.

$$|x+y|_p \leq \max\left\{|x|_p, |y|_p\right\}$$

Proof: Let $x = p^{\text{ord}_p(x)} \frac{m}{n}$ and $y = p^{\text{ord}_p(y)} \frac{s}{t}$ be rational numbers. From the definition of the normed space;

Property 1: It is trivial that; $|x|_p = 0 \Leftrightarrow x = 0$.

Property 2: Since $\text{ord}_p(xy) = \text{ord}_p(x) + \text{ord}_p(y)$; we easily obtain:

$$\|x \cdot y\|_p = \left\|p^{\text{ord}_p(x)} \frac{m}{n} \cdot p^{\text{ord}_p(y)} \frac{t}{s}\right\|_p = \left\|p^{\text{ord}_p(x)} \frac{m}{n}\right\|_p \left\|p^{\text{ord}_p(y)} \frac{t}{s}\right\|_p = \|x\|_p \|y\|_p$$

Property 3: When $x = 0$ or $y = 0$ then it is clear that the triangle inequality holds.

Let $x, y \neq 0$ and $x = \frac{a}{b}$, $y = \frac{c}{d}$. Hence, $x + y = \frac{ad+bc}{bd}$ and

$$\begin{aligned} \text{ord}_p(x+y) &= \text{ord}_p(ad+bc) - \text{ord}_p(bd) \\ &\geq \min\{(\text{ord}_p(ad), \text{ord}_p(bc)) - \text{ord}_p(b) - \text{ord}_p(d)\} \\ &= \min(\text{ord}_p a - \text{ord}_p b, \text{ord}_p c - \text{ord}_p d) \\ &= \min(\text{ord}_p x, \text{ord}_p y) \end{aligned}$$

Therefore; we obtain

$$|x+y|_p = p^{-\mathrm{ord}_p(x+y)} \le \max(p^{-\mathrm{ord}_p x}, p^{-\mathrm{ord}_p y}) = \max(|x|_p, |y|_p) \le |x|_p + |y|_p .$$

This provides that the norm $|\cdot|_p$ satisfies the strong triangle inequality. Therefore; the norm $|\cdot|_p$ is a *non-Archimedean* norm.

Remark 3.3 Let $p_1 \ne p_2$ be prime numbers then $|\cdot|_{p_1} \ne |\cdot|_{p_2}$. Actually, for a sequence $(x_n) = \left(\frac{p_1}{p_2}\right)^n$, $|x_n|_{p_1} \to 0$ but $|x_n|_{p_2} \to \infty$ i.e. $|x_n|_{p_1} = \left|\frac{p_1^n}{p_2^n}\right|_{p_1} = p_1^{-n} \to 0;$ as $n \to \infty$ but $|x_n|_{p_2} = \left|\frac{p_1^n}{p_2^n}\right|_{p_2} = p_2^n \to \infty.$ as $n \to \infty$. Then these norms are different.

Now we are ready to define the *p*-adic numbers field $\mathbf{Q}_p$. Let p be a prime number then we define $\mathbf{Q}_p$ respect to the norm (0.7) as a completion of rational numbers $\mathbf{Q}$. We can expand p-adic norm to the *p*-adic field $\mathbf{Q}_p$; i.e. $\mathbf{Q} \subset \mathbf{Q}_p$. For all $x \in \mathbf{Q}_p$; let $g : x \in \mathbf{Q}_p \to \|x\|_p$, $f : x \in \mathbf{Q} \to f(x)$ and $g(x) = f(x)$ then g is called the expansion of f to the *p*-adic field $\mathbf{Q}_p$ and *p*-adic normed space $\left(\mathbf{Q}_p, |\cdot|_p\right)$ is complete. $\mathbf{Q}_p$ is called p-adic numbers field. Elements of $\mathbf{Q}_p$ are equivalence classes of Cauchy sequences respect to p-adic norm.

Let the sequence (a_n) be a constant Cauchy sequence shown by a for $a \in \mathbf{Q}_p$. Hence, due to definition we write *a* as $a = \sum_{n=k}^{\infty} a_n p^n$ and we get the limit of (a_n) as follows:

$$|a|_p = \lim_{n\to\infty} |a_n|_p \tag{0.8}$$

Range set of *p*-adic norms $\left\{|\cdot|_p\right\}$ which take the same values in $\mathbf{Q}$ and $\mathbf{Q}_p$, i.e. $\left\{p^n,\ n \in \mathbf{Z}\right\} \cup \{0\}$, holds up sufficiently different condition with Euclidean metric. If we expand p-adic norms $|\cdot|_p$ from $\mathbf{Q}$ to $\mathbf{R}$ then they can get all negative values. Let $\forall i > -m$, $0 < \mathrm{d}_{-m} < p$ and $0 \le \mathrm{d}_i < p$, then the following series holds.

$$\frac{d_{-m}}{p^m} + \frac{d_{-m+1}}{p^{m-1}} + \ldots + d_0 + d_1 p + d_2 p^2 + \ldots = \sum_{k=-m}^{\infty} d_k p^k \tag{0.9}$$

Partial sum of (0.9) constructs a Cauchy sequence since for all $\epsilon > 0$, there exists a N and for $p^{-N} < \epsilon$ and $k > n > N$ we get following;

$$\left|\sum_{-m}^{k} d_i p^i - \sum_{-m}^{n} d_i p^i\right|_p = \left|\sum_{n+1}^{k} d_i p^i\right|_p \leq \underset{n<i\leq k}{\text{maks}}(\left|d_i p^i\right|_p) \leq p^{-N} < \epsilon \qquad (0.10)$$

Therefore any sequence (0.9) determines an element of $\mathbf{Q}_p$. This is also conversely true.

Now we need to show that equivalence class of each Cauchy sequence is unique in $\mathbf{Q}$.

To prove this fact, we need the following lemma.

Lemma 3.4 Let $\alpha \in \{0,1,2,..., p^i - 1\}$, If $x \in \mathbf{Q}$ and $|x|_p \leq 1$ then for all i there exists only one $\alpha \in \mathbf{Z}$ that the inequality $|\alpha - x|_p \leq p^{-i}$ holds.

Proof: Let $x = \frac{a}{b}$ where $(a,b) = 1$. Since $|x|_p \leq 1$, p does not divide b and b, p^i are relatively prime. Hence, there exists any two integers m, n that $mb + np^i \equiv 1 \pmod{p}$ holds. Let $\alpha = a \cdot m$, then we get

$$|\alpha - x|_p = \left|am - \frac{a}{b}\right|_p = \left|\frac{a}{b}\right|_p |mb - 1|_p$$
$$\leq |mb - 1|_p = |np^i|_p = |n|_p \, p^{-i} \leq p^{-i}.$$

Finally, impending $|\alpha - x|_p \leq p^{-i}$, we can add a multiple of p^i to α, to obtain an integer between 0 and p^i and still the inequality $|\alpha - x|_p \leq p^{-i}$ holds.

For Instance: Let $x = \frac{10}{21}$ and $p = 5$. It is clear that $(10,21) = 1$. From here, let $(5,21) = 1$. $m = 1$ and $n = 1$ since $\left|\frac{10}{21}\right|_5 = \left|5 \cdot \frac{2}{21}\right|_5 = 5^{-1} \leq 1$, we get $1 \cdot 21 + 1 \cdot 5 \equiv 1 \pmod{5}$. After that for $\alpha = 5 \cdot 1 \equiv 0 \pmod{5}$, we obtain

$$|\alpha - x|_5 = \left|1 \cdot 5 - \frac{10}{21}\right|_5 = \left|\frac{10}{21}\right|_5 |1 \cdot 21 - 1|_5$$
$$\leq |1 \cdot 21 - 1|_5 = |1 \cdot 5|_5 = |1|_5 \, 5^{-1} \leq 5^{-1}$$

and $\overline{0} \in \{0,1,2,3,4\}$ is unique.

Theorem 3.5 In $\mathbf{Q}_p$, there exists one only one (a_i) Cauchy sequence presentation that satisfies the following conditions:

1) $0 \leq a_i \leq p^i - 1$, $a_i \in \mathbf{Z}$, for $i = 1, 2, \ldots$,
2) $a_i \equiv a_{i+1} \pmod{p^i}$, for all $i = 1, 2, \ldots$.

For each equivalence class of *a* which holds the inequality, $|a|_p \leq 1$.

Proof: Let (b_i) be a Cauchy sequence that shows the equivalence class of a. Here our aim is finding a sequence (a_i) which satisfies (1) and (2) and equivalent to the sequence (b_i). We can ignore initial terms of sequence since $|b_i|_p \to |a|_p \leq 1$ as $i \to \infty$. Let $|b_i|_p \leq 1$ for all i and

$|b_i - b_{i'}|_p \leq p^{-j}$, $\forall i, i' \geq N(j)$ holds for all $j = 1, 2, \ldots$, $N(j)$ positive integer. Let us take the sequence $N(j)$ be increasing respect to j; hence $N(j) \geq j$. From Lemma 3.4, for $0 \leq a_j < p^j$, we can find the integers a_j as $|a_j - b_{N(j)}|_p \leq \frac{1}{p^j}$. In order let us show $a_j \equiv a_{j+1} \pmod{p^j}$ and $(b_j) \square (a_j)$. First claim; since

$$\begin{aligned} |a_{j+1} - a_j|_p &= |a_{j+1} - b_{N(j+1)} + b_{N(j+1)} - b_{N(j)} - (a_j - b_{N(j)})|_p \\ &\leq \max\left(|a_{j+1} - b_{N(j+1)}|_p, |-b_{N(j+1)} + b_{N(j+1)}|_p, |a_j - b_{N(j)}|_p\right) \\ &\leq \max\left(\frac{1}{p^{j+1}}, \frac{1}{p^j}, \frac{1}{p^j}\right) = \frac{1}{p^j}. \end{aligned}$$

then we obtain $a_j \equiv a_{j+1} \pmod{p^j}$.

To prove the second claim, let us take a j. For all $i \geq N(j)$; we can obtain,

$$\begin{aligned} |a_i - b_j|_p &= |a_i - a_j + a_j - b_{N(j)} - (b_i - b_{N(j)})|_p \\ &\leq \max\left(|a_i - a_j|_p, |a_j - b_{N(j)}|_p, |b_i - b_{N(j)}|_p\right) \\ &\leq \max\left(\frac{1}{p^j}, \frac{1}{p^j}, \frac{1}{p^j}\right) = \frac{1}{p^j}. \end{aligned}$$

Therefore $|a_i - b_i|_p \to 0$, as $i \to \infty$. This proves the equivalency $(b_j) \square (a_j)$.

Now let us prove the uniqueness: Let us take a sequence (a_i'), and $a_{i_0} \neq a_{i_0}'$ for some i_0. Then, $a_{i_0} \neq a_{i_0}' \pmod{p^{i_0}}$ since $0 < a_{i_0}, a_{i_0}' < p^{i_0}$. Therefore from (2), for $i > i_0$, we obtain $a_i \equiv a_{i_0} \neq a_{i_0}' \equiv a_i' \pmod{p^{i_0}}$ i.e. $a_i \neq a_i' \pmod{p^{i_0}}$. This concludes that $|a_i - a_i'|_p > \frac{1}{p^{i_0}}$ for all $i \geq i_0$. Hence, this proves that the sequences (a_i), (a_i') are not congruent.

From theorem 3.5, every $a \in \mathbf{Q}_p$ can be represented as $a = d_0 + d_1 p + d_2 p^2 + \ldots + d_{i-1} p^{i-1}$ where

$d_i \in \{0,1,2,...,p-1\}$. And this representation is called the canonic form of *a* and it is unique.

Following proposition shows us the uniqueness of the representation of a p-adic norm.

Proposition 3.6 Let $0 \le n < k$ and $d_k \ne 0$. Then

$$|a|_p = \begin{cases} p^{-k}, \text{ whenever } a = \sum_{n=0}^{\infty} d_n p^n, d_n \ne 0, \\ p^m, \text{ whenever, } a = \sum_{n=-m}^{\infty} d_n p^n, d_{-m} \ne 0. \end{cases}$$

Proof: The norm $|a|_p$ is the limit of the partial sum of the p-adic norms of sequences. In the first case: Since $|d_k|_p = 1$ and from strong triangle inequality, we get constants sequence $p^{-k}, p^{-k}, p^{-k},...$. Hence, $|a|_p = p^{-k}$. If $a = \sum_{n=-m}^{\infty} d_n p^n$ and $d_{-m} \ne 0$ then by the same process we obtain, $|a|_p = p^m$.

Definition 3.7 If the canonic expansion of any $a \in \mathbf{Q}_p$ includes only non-negative powers of *p* then it is called *p*-adic integer. And the set of p-adic integers denoted by $\mathbf{Z}_p$. It is clear that

$$\mathbf{Z}_p = \left\{ x \in \mathbf{Q}_p : x = \sum_{i=0}^{\infty} a_i p^i \right\}.$$

From here we define *p*-adic integers set $\mathbf{Z}_p$, as follows:

$$\mathbf{Z}_p = \left\{ a \in \mathbf{Q}_p : |a|_p \le 1 \right\}.$$

Moreover, the set of *p*-adic units is denoted by $\mathbf{Z}_p^{\times}$ and defined as follows:

$$\mathbf{Z}_p^{\times} = \left\{ x \in \mathbf{Z}_p : |x|_p = 1 \right\}.$$

Theorem 3.8 Each p-adic integer sequence has a convergent subsequence.

Proof: Let us remind the subsequences (x_{n_k}) of positive integer sequences (x_k) s.t. $n_1 < n_2 < n_3 < ....$ Let $(x_k) \in \mathbf{Z}_p$ and canonic expansion of each term is $x_k = ...a_2^k a_1^k a_0^k$. For the digit a_0^k, since the existence of finite variations we can find infinity subsequence $(\mathbf{x}_{0_k})$ with the last digit $b_0 \in \{0,1,2,...,p-1\}$, of the finite digit sequence (x_k). If we continue this process with $b_0, b_1, b_2, ...$ we can get the following sequence;

$$x_{00}, x_{01}, x_{02}, ..., x_{0s}, ...$$
$$x_{10}, x_{11}, x_{12}, ..., x_{1s}, ...$$
$$x_{20}, x_{21}, x_{22}, ..., x_{2s}, ...$$
$$...............................$$

Here, each sequence is a subsequence of next one and each element of n^{th} row ends with $b_n ... b_1 b_0$. let $, x_{jj} \in \{x_{j-1j}, x_{j-1j+1}, ...\}$, for all $j = 0,1,...$. Then the diagonal sequence $x_{00}, x_{11}, ...$ is still subsequence of original sequence, and it is clear this subsequence converges $...b_3 b_2 b_1 b_0$.

3.1 p-Adic Expansion of Rational Numbers

Each rational integer is also *p*-adic integer i.e. each rational integer can be expressed as an expansion of base *p*. For instance, the number -1 can be expressed as $-1 = (p-1)\sum_{i=0}^{\infty} p^i$. Then we can express any *p*-adic integer in $\mathbf{Z}_p$ as $\sum_{i=0}^{\infty} p^i = \frac{1}{1-p}$, $\frac{1}{1-p} = ...1111$.

Next theorem provides that each rational number can be expressed as an expansion of *p*-adic numbers.

Theorem 3.8 Let $a = ...d_n ... d_2 d_1 d_0 . d_{-1} ... d_{-m}$ be a rational number. If the *p*-adic expansion of *a* is left-hand periodic then this *p*-adic expansion states a rational number.

Proof: $\Rightarrow$: We can multiply a p-adic number x by the power of *p* and then we can subtract a rational number from the result. Any p-adic integer $x \in \mathbf{Z}_p$ can be expressed as follows:

$$x = x_0 + x_1 p + x_2 p^2 + ... + x_{k-1} p^{k-1} + x_0 p^k + x_1 p^{k+1} + ...$$

And let $a = x_0 + x_1 p + x_2 p^2 + ... + x_{k-1} p^{k-1}$ be a rational number, x can be stated as follows

$$x = a(1 + p^k + p^{2k} + ... = a \cdot \frac{1}{1-p^k}.$$

Then x is a rational.

$\Leftarrow$: Assume that the following p-adic expansion is not a rational number.

$$\frac{a}{b} = \sum_{i \ge 0} x_i p^i \in \mathbf{Z}_p \tag{0.11}$$

Suppose that $(a,b) = 1$ and $(b, p) = 1$. Since $(b, p^n) = 1$, there exists such c_n, d_n that the equation $c_n b + d_n p^n = 1$ holds. We get $a = ac_n b + ad_n p^n$ by multiplying *a*. When we add a multiple of p^n to ac_n, we get any two integers A_n and r_n where $0 \le A_n \le p^n - 1$. Hence, $a = A_n b + r_n p^n$ holds. If we divide both sides of the previous equation by *b* then we obtain

$$\frac{a}{b} = A_n + \frac{r_n}{b} p^n$$

Then

$$\frac{a-(p^n-1)b}{p^n} \le r_n \le \frac{a}{p^n}$$

since $r_n = (a - A_n b)/p^n$. For sufficient big n, it becomes $-b \le r_n \le 0$. This means that r_n is finite valuable. Therefore, we can express $\frac{a}{b}$ as follows:

$$\frac{a}{b} = A_n + p^n \frac{r_n}{b} = A_{n+1} + p^{n+1}\frac{r_{n+1}}{b} \quad (0.12)$$

$\frac{r_n - pr_{n+1}}{b}$ is an integer since $A_{n+1} - A_n = p^n\left(\frac{r_n - pr_{n+1}}{b}\right)$ is an integer and $(b, p^n) = 1$. From $A_{n+1} \equiv A_n (\operatorname{mod} p^n)$ and Theorem 3.5, the sequence (A_n) is the partial sum of p-adic canonic presentation of $\frac{a}{b}$. For all n's, $A_{n+1} = A_n + x_n p^n$ and from the assumption (0.11) we obtain $r_n = x_n b + pr_{n+1}$. Since r_n has finite valuable sequence, there exists such an index m and a positive integer P that $r_m = r_{m+P}$. Hence we get

$$x_m b + pr_{m+1} = x_{m+P} b + pr_{m+P+1} \quad (0.13)$$

Then p divides $x_m - x_{m+P}$ since $(b, p) = 1$ in the equation $(x_m - x_{m+P})b = p(r_{m+P+1} - r_{m+1})$. But from $x_m, x_{m+P} \in \{0,1,..., p-1\}$ we get $x_m = x_{m+P}$. If we substitute this in (0.13) we conclude that $r_{m+1} = r_{m+P+1}$. If we use same processes then we find $r_n = r_{n+P}$ and $x_n = x_{n+P}$ $(n \ge m)$. This proves the existence P -long period in both x_n and r_n for $n \ge m$.

4. Conclusion

In the present work, we gave the definition of the p-adic numbers field and some of its important analysis with proofs. Moreover, we showed some analogy between real numbers and p-adic numbers and some important differences such as ultra-metric triangles.

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