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# ON THE SPECTRUM OF NORLUND TYPE MATRIX OPERATOR $A = (a_{nk})$ ON THE SEQUENCE SPACES $\ell_1$ AND $bv$

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**Abstract:**

In this article, we defined a Nörlund type matrix $A = (a_{nk})$ by

$$a_{nk} = \begin{cases} 1 & , \quad k = n = 0 \\ \frac{1}{2} & , \quad n-1 \leq k \leq n. \\ 0 & , \quad othervise \end{cases}$$

Then we showed that the Nörlund type matrix $A = (a_{nk})$ is a linear and bounded operator on the sequence spaces $\ell_1$ and $bv$. Finally, we calculated the fine spectrum and its subdivisions of the operator $A = (a_{nk})$ on the sequence spaces $\ell_1$ and $bv$.

**Keywords:** *Sequence Space; Bounded Variation; Nörlund Type Matrix, Bounded Operators; Spectrum of an Operator.*

## 1. Introduction

By $\omega$, we represent the set of all complex-valued sequences. We call each vector subspace of $\omega$ as a sequence space. The sets $\ell_\infty$, $c$ and $c_0$ are bounded, convergent and null sequence spaces, respectively. The set $\phi$ represents all finitely non-zero called sequences. If a sequence space $\mu$ is a complete linear metric space with continuous coordinates $p_n: \mu \to \mathbb{C}$ with $p_n(x) = x_n$ for all $x = (x_n) \in \mu$ and every $n \in \mathbb{N}$, then it is called an $FK$-space, where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0,1,2,\ldots\}$.

If a sequence space $\lambda$ is an $FK$-space, then we say that $\phi \subset \lambda$ and the sequence $(e^k)$ is a basis for the sequence space $\lambda$, where $e^k$ is a sequence whose only term in $k^{th}$ place is 1 the others are zero for each $k \in \mathbb{N}$ and $\phi = span\{e^k\}$. Thus, we say that $\lambda$ has $AK$ property. Moreover, $\lambda$ is named $AD$-space if the set $\phi$ is dense in $\lambda$. Thus $AK$ implies $AD$.

A normed $FK$-space is called a $BK$-space. Therefore, a $BK$-space is a Banach space with continuous coordinates, [1, pp. 272-273]. The classical sequence spaces $\ell_\infty$, $c$ and $c_0$ are $BK$-spaces with the sup-norm defined by

$$\| x \|_{\infty} = \sup_{k\in\mathbb{N}} |x_k|.$$

The space of absolutely summable sequences denoted by $\ell_1$ is defined by

(1) $$\ell_1 = \{x = (x_k) \in \omega : \textstyle\sum_{k=0}^{\infty} |x_k| < \infty\},$$

and it is a $BK$-space with the folowing norm

$$\| x \|_{\ell_1} = \sum_{k=0}^{\infty} |x_k| < \infty.$$

Moreover, the space of all sequences of bounded variation denoted by $bv$ is defined by

(2) $$bv = \{x = (x_k) \in \omega : \textstyle\sum_{k=0}^{\infty} |x_k - x_{k-1}| < \infty\},$$

and it is a $BK$-space with the folowing norm

$$\| x \|_{bv} = |x_0| + \sum_{k=1}^{\infty} |x_k - x_{k-1}| < \infty.$$

On one hand, the sequence space $bv$ is defined as the backward difference operator $\Delta$ domain on the sequence space $\ell_1$, where $\Delta x_k = x_k - x_{k-1}$ for all $k \in \mathbb{N}$. On the other hand, we can also represent $bv$ as

(3) $$bv = \{x = (x_k) \in \omega : \textstyle\sum_{k=1}^{\infty} |x_k - x_{k+1}| < \infty\},$$

Which is the forward difference operator $\Delta$ domain on the sequence space $\ell_1$, where $\Delta x_k = x_k - x_{k+1}$, for all $k \in \mathbb{N}$. The space $bv_0 = bv \cap c_0$ and the inclusions $\ell_1 \subset bv_0 \subset bv \subset c$ are strictly hold.

The beta-dual $\lambda^{\beta}$ of a sequence space $\lambda$ is the set of all $x = (x_k) \in \omega$ such that $xy = (x_k y_k) \in cs$ for all $y = (y_k) \in \lambda$. Let $\lambda$, $\mu$ be any two subsets of $\omega$ and $A = (a_{nk})_{n,k=1}^{\infty}$ be an infinite matrix of complex numbers. Then, we have for $x = (x_k)$

$$A_n x = \sum a_{nk} x_k \quad and \quad Ax = (A_n x)_{n=1}^{\infty}$$

Provided all the series converge, the set $\lambda_A = \{x \in \omega : Ax \in \lambda\}$ is called the matrix domain of $A$ in $\lambda$, and $(\lambda, \mu)$ denotes the class of all matrix transformations from $\lambda$ into $\mu$. Therefore, $A \in (\lambda, \mu)$ if and only if $\lambda \subset \mu_A$, $A_n \in \lambda^{\beta}$ for all $n \in \mathbb{N}$, and $A_n x$ belongs to $\mu$ for all $x \in \lambda$, where $A_n$ denotes the $n$-th row of $A$.

## 2. Spectrum of Bounded Operators

Let $X$ be a linear space. The set of all linear operators $T: X \to X$ is denoted by $L(X)$. By $B(X)$, we denote the set of all bounded linear operators on $X$ into itself. Suppose that $X$ is a Banach space. Then, if $T \in B(X)$, the adjoint operator $T^*$ of $T$ is in $B(X^*)$, where $X^*$ is dual space of $X$, defined by

(4) $$(T^* y^*)(x) = y^*(Tx) \;\; for\ all \;\; y^* \in X^* \;\; and \;\; x \in X.$$

Let $T: D(T) \to X$ be a linear operator on its domain $D(T) \subset X$ into the complex normed space $X$. For $T \in B(X)$ we associate a complex number $\lambda$ with the operator $(T - \lambda I)$ denoted by $T_{\lambda}$ which is called the perturbed operator on the same domain $D(T)$ as $T$ where $I$ denotes the identity operator. The inverse $(T-\lambda I)^{-1}$, which is denoted by $T_{\lambda}^{-1}$ is called the resolvent operator of $T_{\lambda}$. Regarding the detailed information about the spectrum and its subdivisions we refer authors to the referee [2, Proposition 1.2, 1.3, p. 27-28], [10], [12], [19], and [20].

## 3. The Spectrum of Nörlund Type Matrix Operator $A=(a_{nk})$ on the Sequence Spaces $\ell_1$ and $bv$

In this section, we calculate the fine spectrum and its subdivisions of the matrix operator $A=(a_{nk})$ on the sequence spaces $\ell_1$ and $bv$. First, we start with the known facts about the Nörlund mean and its inverse.

The Nörlund mean with respect to the non-negative sequence $t=(t_k)$, where $t_0>0$ and $T_n=\sum_{k=0}^{n} t_k$ for all $n\in\mathbb{N}$, is defined in the matrix form $N^t=(a_{nk}^t)$ by

$$a_{nk}^t=\begin{cases}\frac{t_{n-k}}{T_n} & , \quad 0\le k\le n,\\ 0 & , \quad k>n\end{cases}$$

For every $k,n\in\mathbb{N}$. Moreover, let $D_n$, where $t_0=D_0=1$, is defined for each $n\in\{1,2,3,\ldots\}$ as follows;

$$D_n=\begin{vmatrix} t_1 & 1 & 0 & 0 & \cdots & 0\\ t_2 & t_1 & 1 & 0 & \cdots & 0\\ t_3 & t_2 & t_1 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ t_{n-1} & t_{n-2} & t_{n-3} & t_{n-4} & \cdots & 1\\ t_n & t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_1\end{vmatrix}.$$

Then, Mears [4] defined the inverse $U^t=(u_{nk}^t)$ of Nörlund matrix $N^t$ as follows:

$$u_{nk}^t=\begin{cases}(-1)^{n-k}D_{n-k}T_k & , \quad 0\le k\le n\\ 0 & , \quad k>n\end{cases}.$$

Moreover, the inverse of Nörlund matrix and some multiplication theorems for Nörlund mean were studied by Mears [5, 4]. The Nörlund matrix domain in some sequence spaces were calculated by some distinguised mathematicians (see [6, 7, 8, 9]) as an application. Moreover, we would like to direct readers to the following refrees (see [11, 12, 13]) to see the potential studies on the spectral results of Nörlund mean and some special Nörlund matrices as a bounded and linear operator on some sequence spaces.

In this paper, we calculate the spectrum of a special Nörlund matrix $A=(a_{nk})$ on the sequence spaces $\ell_1$ and $bv$. We begin with defining the Nörlund type matrix operator $A=(a_{nk})$, and then we show that $A=(a_{nk})$ is linear and bounded operator on the sequence spaces $\ell_1$ and $bv$. The main body of the paper presents the spectrum of the Nörlund type matrix $A=(a_{nk})$ and its subdivisions on the sequence spaces $\ell_1$ and $bv$.

Let the Norlund type matrix $A=(a_{nk})$ is defined as follows. $A=(a_{nk})$ by

$$a_{nk}=\begin{cases}1 & , \quad k=n=0\\ \dfrac{1}{2} & , \quad n-1\le k\le n\\ 0 & , \quad otherwise\end{cases}$$

Equivalently

$$a_{nk}=\begin{bmatrix}1 & 0 & 0 & 0 & \cdots\\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots\\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots\\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \cdots\\ \vdots & \vdots & \vdots & \ddots & \ddots\end{bmatrix}.$$

**Lemma 3.1** The matrix $A=(a_{nk})$ is a bounded linear operator, $A\in B(\ell_1)$ from $\ell_1$ to itself, if and only if the supremum of $\ell_1$ norms of the columns of $A$ is bounded, i.e.,

$$\sup_{k\in\mathbb{N}}\sum_{n=1}^{\infty}|a_{nk}|<\infty.$$

**Lemma 3.2** The matrix $A=(a_{nk})$ is a bounded linear operator, $A\in B(bv)$ from $bv$ to itself, if and only if

$$\sup_{k}\sum_{n=0}^{\infty}\left|\sum_{i=k}^{\infty}(a_{ni}-a_{n-1,i})\right|<\infty.$$

Then the following holds:

**Theorem 3.1** The infinite matrix $A\in(\ell_1,\ell_1)$ if and only if the condition in (17) holds for $A=(a_{nk})$ defined by (16).

**Proof.** We apply the Lemma 3.1 to see the result as in the following

$$\sup_{k\in\mathbb{N}}\sum_{n=0}^{\infty}|a_{nk}|=\begin{cases}\dfrac{3}{2} & , \quad if k=0,\forall n\in\mathbb{N}\\ 1 & , \quad if k\ge 1,\forall n\in\mathbb{N}\end{cases}.$$

Threfore $\sup_{k\in\mathbb{N}}\sum_{n=0}^{\infty}|a_{nk}|\le\frac{3}{2}$.

The following result is cleally seen since it’s proof is similar to the proof of Theorem 3.1 by using Lemma 3.2.

**Corollary 3.1** The infinite matrix $A\in(bv,bv)$ if and only if

$$\sup_{k}\sum_{n=0}^{\infty}\left|\sum_{i=k}^{\infty}(a_{ni}-a_{n-1,i})\right|\le\frac{3}{2}.$$

**Theorem 3.2** The spectrum of the matrix operator $A=(a_{nk})$ on the sequence space $\ell_1$ is the set

$$\sigma(A,\ell_1)=\left\{\lambda\in\mathbb{C}:\left|\tfrac{1}{2}-\lambda\right|\leq\tfrac{1}{2}\right\}.$$

**Proof.** Suppose that $|\frac{1}{2}-\lambda|>\frac{1}{2}$. Since the matrix $A$ is triangular, then $(A-\lambda I)^{-1}=B=(b_{nk})$ is given by

$$(b_{nk})=\begin{bmatrix}1-\lambda & 0 & 0 & \cdots\\ \frac{1}{2} & \frac{1}{2}-\lambda & 0 & \cdots\\ 0 & \frac{1}{2} & \frac{1}{2}-\lambda & \cdots\\ \vdots & \vdots & \ddots & \ddots\end{bmatrix}^{-1}=\begin{bmatrix}\frac{1}{1-\lambda} & 0 & 0 & \cdots\\ \frac{1}{1-\lambda}\left(\frac{-1}{(1-2\lambda)}\right) & \frac{2}{1-2\lambda} & 0 & \cdots\\ \frac{1}{1-\lambda}\left(\frac{1}{(1-2\lambda)^2}\right) & \frac{-2}{(1-2\lambda)^2} & \frac{2}{1-2\lambda} & \cdots\\ \vdots & \vdots & \ddots & \ddots\end{bmatrix}$$

Where

$$b_{nk}=\begin{cases}\frac{1}{1-\lambda}(-1)^{k-n}\frac{1}{(1-2\lambda)^{k-n}} & ,\quad n=0,\forall k\in\mathbb{N}\\ (-1)^{k-n}\frac{2}{(1-2\lambda)^{k-n+1}} & ,\quad n,k\geq 1,\\ 0 & ,\quad k>n\end{cases}$$

The aim of us here is to show that the matrix $B$ is a bounded linear operator on $\ell_1$. We need to apply Lemma 3.1 for the matrix $B$. Therefore, suppose that a natural number $m$ is given such that $m\geq k$ for every $k\in\mathbb{N}$ such that

$$\sum_{n=0}^{\infty}|a_{nk}|=\begin{cases}\left|\frac{1}{1-\lambda}\right|\left(\frac{1-\left(\frac{1}{|1-2\lambda|}\right)^{m+1}}{1-\frac{1}{|1-2\lambda|}}\right) & ,\quad if n=0,\forall k=0,1,2,3,\ldots,m,\\ \left|\frac{2}{1-2\lambda}\right|\left(\frac{1-\left(\frac{1}{|1-2\lambda|}\right)^{m-1}}{1-\frac{1}{|1-2\lambda|}}\right) & ,\quad if n\geq 1,\forall k=0,1,2,3,\ldots,m.\end{cases}$$

Now, if we get supremum over $k\in\mathbb{N}$ then we have

$$\sup_{k\in\mathbb{N}}\sum_{n=0}^{\infty}|a_{nk}|<\infty$$

Since $|\frac{1}{2}-\lambda|>\frac{1}{2}$. It completes the proof.

**Theorem 3.3** The point spectrum $\sigma_p(A)$ of the matrix $A$ on the sequence space $\ell_1$ is the set

$$\sigma_p(A,\ell_1)=\emptyset.$$

**Proof.** Suppose that the characteristic equation $Ax=\lambda x$ for all corresponding eigenvector $x\neq 0$ of eigenvalues $\lambda\in\mathbb{C}$ in the sequence space $\ell_1$. Let us calculate the equation $Ax=\lambda x$, where $x\neq 0$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} = \lambda \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix}$$

Which gives the following equations

$$x_0 = \lambda x_0$$

$$\frac{1}{2}x_0 + \frac{1}{2}x_1 = \lambda x_1$$

$$\frac{1}{2}x_1 + \frac{1}{2}x_2 = \lambda x_2$$

If $\lambda = 1$ for the first nonzero eigenvector $x_0$, then $x_0 = x_1 = x_2 = \ldots$ which says that $x$ is in the span of $\delta = (1,1,1,\ldots)$ which tends to 1 as $(n \to \infty)$. Therefore, $x \notin \ell_1$ and $\lambda = 1$ is not an eigenvalue of $A \in B(\ell_1)$. Moreover, if $x_0 = 0$ and $x_1$ is the first nonzero entry of $x$, then $\lambda = \frac{1}{2}$. But in this case $x_0 = x_1 = x_2 = \ldots = 0$ which is a contradiction. Therefore, $\lambda = \frac{1}{2}$ is not an eigenvalue of $A \in B(\ell_1)$. It completes the proof.

The adjoint $T^*$ of a linear operator $T \in B(X,Y)$ is the mapping from $Y^*$ to $X^*$ defined by $T^* \circ f = f \circ T$, where $f \in Y^*$. Moreover, since $T$ is linear and bounded, $T^*$ is linear and bounded, and $\| T^* \| = \| T \|$ (see [15]).

If $A \in B(\ell_1)$ with a matrix $A$, then the adjoint operator $A^*: \ell_1^* \to \ell_1^*$ is defined by the transpose $A^t$ of the matrix $A$. So we should note here that the dual space $\ell_1^*$ of $\ell_1$ is isometrically isomorphic to $\ell_\infty$ of bounded sequences which is normed by $\| x \| = \sup_n |x_n|$. Then we have the following theorems and results.

**Theorem 3.4** $A \in (\ell_\infty; \ell_\infty)$ if and only if

$$\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty.$$

**Theorem 3.5** Let $A: \ell_1 \to \ell_1$. Then $A^*: \ell_\infty \to \ell_\infty$ if and only if

$$\sup_n \sum_{k=0}^{\infty} |a_{nk}^*| < \infty.$$

**Theorem 3.6** $\sigma_p(A^*, \ell_1) = \left\{\lambda \in \mathbb{C}: \left|\lambda - \frac{1}{2}\right| < \frac{1}{2}\right\}$

**Proof.** We solve the following equation which is $\mathrm{A^*x = \lambda x}$, where $\mathrm{x} \in \ell_\infty$, $\mathrm{x_k} \neq 0, \forall \mathrm{k} \in \mathbb{N}, \lambda \in \mathbb{C}$ that

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} = \lambda \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix}.$$

Which gives the following system of equation

$$x_0 + \frac{1}{2}x_1 = \lambda x_0 x_1 = 2(\lambda - 1)x_0$$

$$\frac{1}{2}x_1 + \frac{1}{2}x_2 = \lambda x_1 x_2 = 2(\lambda - \frac{1}{2})x_1 = \left[2\left(\lambda - \frac{1}{2}\right)\right] x_1$$

$$\frac{1}{2}x_2 + \frac{1}{2}x_3 = \lambda x_2 x_3 = 2(\lambda - \frac{1}{2})x_2 = \left[2\left(\lambda - \frac{1}{2}\right)\right]^2 x_1$$

$$\vdots$$

$$\frac{1}{2}x_{n-1} + \frac{1}{2}x_n = \lambda x_{n-1} x_n = 2(\lambda - \frac{1}{2})x_{n-1} = \left[2\left(\lambda - \frac{1}{2}\right)\right]^{n-1} x_1$$

$$\vdots.$$

Therefore, the solution of the above system of equation gives that

$$x_n = \left[2\left(\lambda - \frac{1}{2}\right)\right]^{n-1} x_1, \qquad \forall n \in \mathbb{N}$$

Now we should show that $x = (x_n) \in \ell_\infty$, that is,

$$\sup_{n\in\mathbb{N}} |x_n| = \sup_{n\in\mathbb{N}} \left| \left[2\left(\lambda - \frac{1}{2}\right)\right]^{n-1} x_1 \right| = |x_1| \sup_{n\in\mathbb{N}} \left(2\left|\lambda - \frac{1}{2}\right|\right)^{n-1} < \infty$$

Since $2\left|\lambda - \frac{1}{2}\right| < 1$. It completes the proof.

**Lemma 3.3** [16, p. 59] An operator $T$ has dense range if and only if $T^*$ is one to one.

**Lemma 3.4** [16, p. 60] The adjoint operator $T^*$ of $T$ is onto if and only if $T$ has a bounded inverse.

**Theorem 3.7** The resudial spectrum $\sigma_r(A, \ell_1)$ of the operator $A$ on the sequence space $\ell_1$ is the set $\left\{\lambda \in \mathbb{C}: \left|\frac{1}{2} - \lambda\right| \leq \frac{1}{2}\right\}$, that is

$$\sigma_r(A, \ell_1) = \left\{\lambda \in \mathbb{C}: \left|\frac{1}{2} - \lambda\right| \leq \frac{1}{2}\right\}$$

**Proof.** Suppose that $\left|\frac{1}{2}-\lambda\right| \leq \frac{1}{2}$. We need to show that $A-\lambda I$ has an inverse and $\overline{B}(A-\lambda I) \neq \ell_1$ for $\lambda$ which stisfied $\left|\frac{1}{2}-\lambda\right| \leq \frac{1}{2}$. Suppose that $\lambda \neq \frac{1}{2}$. Then the operator $A-\lambda I$ is a triangle hence has a unique inverse. Moreover, suppose that $\lambda = \frac{1}{2}$. Then the operator $\left(A-\frac{1}{2}I\right)$ has the representation as

$$\left(A-\frac{1}{2}I\right) = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Hence $\left(A-\frac{1}{2}I\right)x = \theta$ implies $x = \theta$, $A-\frac{1}{2}I: \ell_1 \to \ell_1$ is injective but has an inverse. But, the operator $A^*-\frac{1}{2}I$ which is represented as

$$\left(A^*-\frac{1}{2}I\right) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

Which is not one to one. Therefore, we can say by Lemma 3.3 that the range of the operator $A-\frac{1}{2}I$ is not dense in the sequence space $\ell_1$. It completes the proof.

**Corollary 3.2** Continuous spectrum of the matrix $A$ on the sequence space $\ell_1$ is the empty set, that is, $\sigma_c(A,\ell_1) = \emptyset$.

**Proof.** By considering the Theorem 3, Theorem 3 and Theorem 3, and since $\sigma(A,\ell_1)$ is the disjoint union of the subdivisions $\sigma_p(A,\ell_1)$, $\sigma_r(A,\ell_1)$, and $\sigma_c(A,\ell_1)$, we can easily see that $\sigma_c(A,\ell_1) = \emptyset$. It completes the proof.

**Theorem 3.8** The spectrum of the matrix operator $A = (a_{nk})$ on the sequence space $bv$ is the set

$$\sigma(A,bv) = \left\{\lambda \in \mathbb{C}: \left|\frac{1}{2}-\lambda\right| \leq \frac{1}{2}\right\}.$$

**Proof.** Suppose that $|\frac{1}{2}-\lambda| > \frac{1}{2}$. Since the matrix $A$ is triangular, then $(A-\lambda I)^{-1} = B = (b_{nk})$ is given by

$$(b_{nk}) = \begin{bmatrix} 1-\lambda & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2}-\lambda & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2}-\lambda & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{1-\lambda} & 0 & 0 & \cdots \\ \frac{1}{1-\lambda}\left(\frac{-1}{(1-2\lambda)}\right) & \frac{2}{1-2\lambda} & 0 & \cdots \\ \frac{1}{1-\lambda}\left(\frac{1}{(1-2\lambda)^2}\right) & \frac{-2}{(1-2\lambda)^2} & \frac{2}{1-2\lambda} & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

Where

$$b_{nk} = \begin{cases} \frac{1}{1-\lambda}(-1)^{k-n}\frac{1}{(1-2\lambda)^{k-n}} & , \quad n = 0, \forall k \in \mathbb{N} \\ (-1)^{k-n}\frac{2}{(1-2\lambda)^{k-n+1}} & , \quad n, k \geq 1, \\ 0 & , \quad k > n \end{cases}$$

The aim of us here is to show that the matrix $B$ is a bounded linear operator on $bv$. We need to apply Lemma 3.2 for the matrix $B$. Therefore, suppose that a natural number $m$ is given such that $m \geq n$ for every $n \in \mathbb{N}$ such that

$$S_m = \sum_{n=0}^{m}\left|\sum_{i=k}^{\infty}(a_{ni} - a_{n-1,i})\right| = \begin{cases} \frac{1}{|1-\lambda|} & , \quad if n = 0, \forall k \in \mathbb{N} \\ 0 & , \quad if n = 1, \forall k \in \mathbb{N} \\ \frac{4|1-\lambda|}{|1-2\lambda|}\left(\frac{1}{|1-2\lambda|}\right)^{m-k} & , \quad if m \geq n \geq 2, \forall k \in \mathbb{N}. \end{cases}$$

Now, if we first get limit as $m \to \infty$ and then supremum over $k \in \mathbb{N}$, then we have

$$\sup_k \sum_{n=0}^{\infty}\left|\sum_{i=k}^{\infty}(a_{ni} - a_{n-1,i})\right| < \infty$$

Since $|1 - 2\lambda| > 1$. It completes the proof.

**Theorem 3.9** The point spectrum $\sigma_p(A, bv)$ of the matrix $A$ on the sequence space $bv$ is the set

$$\sigma_p(A, bv) = \{1\}.$$

**Proof.** Suppose that the characteristic equation $Ax = \lambda x$ for all corresponding eigenvector $x \neq 0$ of eigenvalues $\lambda \in \mathbb{C}$ in the sequence space $\ell_1$. Let's calculate the equation $Ax = \lambda x$, where $x \neq 0$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} = \lambda\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix}$$

Which gives the following equations

$$x_0 = \lambda x_0$$

$$\frac{1}{2}x_0+\frac{1}{2}x_1=\lambda x_1$$

$$\frac{1}{2}x_1+\frac{1}{2}x_2=\lambda x_2$$

$$\vdots$$

If $\lambda = 1$ for the first nonzero eigenvector $x_0$, then $x_0 = x_1 = x_2 = ..$. which says that $x$ is in the span of $\delta = (1,1,1,\ldots)$ which tends to 1 as $(n \to \infty)$. Therefore, $x \in bv$ and $\lambda = 1$ is an eigenvalue of $A \in B(bv)$.

Moreover, if $x_0 = 0$ and $x_1$ is the first nonzero entry of $x$, then $\lambda = \frac{1}{2}$. But in this case $x_0 = x_1 = x_2 = \ldots = 0$ which is a contradiction. Therefore, $\lambda = \frac{1}{2}$ is not an eigenvalue of $A \in B(bv)$. It completes the proof.

**Theorem 3.10** Let $T: bv \to bv$ be a bounded linear operator with the matrix $A = (a_{nk})$. Then we define $T^*: bv^* \to bv^*$ where $bv^*$ is acting on $\mathbb{C} \oplus bs$ which has the following matrix representation

$$A^* = \begin{pmatrix} \overline{\chi(A)} & (a_k)_{k=0}^{\infty} - \overline{\chi(A)} \\ (b_n)_{n=0}^{\infty} & (a_{kn} - b_n)_{n=0}^{\infty} \end{pmatrix}$$

Where $\overline{\chi(A)} = \lim_n \sum_{k=0}^{\infty} a_{nk}$, $b_n = \lim_{k\to\infty} a_{kn}$, $a_k = P_k(T(\delta))$ where $\delta = (1,1,1,\ldots)$ and $P_k$ is the $k^{th}$ $-$coordinate function for each $k \in \mathbb{N}$ (see [17]).Therefore, for the operator $A: bv \to bv$, the adjoint matrix $A^* \in (\mathbb{C} \oplus bs)$ is calculated as the matrix

$$A^* = \begin{pmatrix} \overline{\chi(A)} & (a_k)_{k=0}^{\infty} - \overline{\chi(A)} \\ (b_n)_{n=0}^{\infty} & (a_{kn} - b_n)_{n=0}^{\infty} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

$$\sigma_p(A^*, bv^*) = \left\{\lambda \in \mathbb{C}: \left|\lambda - \frac{1}{2}\right| \le \frac{1}{2}\right\}$$

**Proof.** We solve the following equation which is $\mathrm{A}^*\mathrm{x} = \lambda\mathrm{x}$, where $\mathrm{x} \in \mathbb{C} \oplus \mathrm{bs}$, $\mathrm{x_k} \neq 0, \forall \mathrm{k} \in \mathbb{N}, \lambda \in \mathbb{C}$ that

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} = \lambda \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix}$$

Which gives the following system of equation

$$x_0 = \lambda x_0$$

$$x_1 + \frac{1}{2}x_2 = \lambda x_1 x_2 = 2(\lambda - 1)x_1$$

$$\frac{1}{2}x_2 + \frac{1}{2}x_3 = \lambda x_2 x_3 = 2\left(\lambda - \frac{1}{2}\right)x_2 = 4(\lambda - 1)\left(\lambda - \frac{1}{2}\right)x_1$$

$$\frac{1}{2}x_{n-1} + \frac{1}{2}x_n = \lambda x_{n-1}x_n = 2\left(\lambda - \frac{1}{2}\right)x_{n-1} = 2(\lambda - 1)\left[2\left(\lambda - \frac{1}{2}\right)\right]^{n-2} x_1$$

Therefore, the solution of the above system of equation gives that

$$x_n = 2(\lambda - 1)\left[2\left(\lambda - \frac{1}{2}\right)\right]^{n-2} x_1, \qquad \forall n \geq 2$$

If $\lambda = 1$, then $x_2 = 0, x_3 = 0, \dots, x_n = 0, \dots$ so that $x = (x_0, x_1, 0,0,0, \dots)$ is an eigenvector corresponding to the eigenvalue $\lambda = 1$. If $\lambda \neq 1$, then $x \in \mathbb{C} \bigoplus bs$ if and only if

$$\sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{n-2} 2(\lambda - 1)\left(2\left(\lambda - \frac{1}{2}\right)\right)^{n-2} x_1 \right| \leq 2|x_1||\lambda - 1| \frac{1 - \left(2\left(\lambda - \frac{1}{2}\right)\right)^{n-1}}{1 - 2\left(\lambda - \frac{1}{2}\right)} < \infty$$

If and only if $2\left|\lambda - \frac{1}{2}\right| < 1$. It completes the proof.

**Corollary 3.3** By using [2, Proposition 1.2 (iii), p. 27] and [2, Proposition 1.3 (v), p.28], we have the following results.

(i) $\sigma_{co}(A, bv) = \left\{\lambda \in \mathbb{C}: \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}$.

(ii) $\sigma_r(A, bv) = \emptyset$.

(iii) $\sigma_c(A, bv) = \emptyset$.

**Corollary 3.4** By using [2, Proposition 1.2, p27] and [2, Proposition 2.3, p.28], we also can state the following results.

(i) $\sigma_{co}(A, \ell_1) = \left\{\lambda \in \mathbb{C}: \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}$.

(ii) $\sigma_{ap}(A,\ell_1)=\left\{\lambda\in\mathbb{C}:\left|\lambda-\frac{1}{2}\right|=\frac{1}{2}\right\}$.

(iii) $\sigma_{\delta}(A^*,\ell_\infty)=\left\{\lambda\in\mathbb{C}:\left|\lambda-\frac{1}{2}\right|=\frac{1}{2}\right\}$.

(iv) $\sigma_{ap}(A,bv)=\emptyset$.

(v) $\sigma_{\delta}(A^*,bv)=\emptyset$.

## 5. Conclusion

Spectral studies are commonly studied problems of bounded and linear operators on sequence spaces. Some significant papers have been studied where the spectral results took place on the absolutely summable sequence space $\ell_1$ and the sequence space $bv$ of bounded variation (see [18, 19, 20]). Moreover, a very recent paper on the fine spectrum of generalized difference operator $\Delta_i^3$ on the Hahn sequence space $h$, which is included by $\ell_1$ and $bv$, has been studied by Malkowsky at al [10]. Moreover, some spectral studies of the Nörlund type matrix on some sequence space have also been calculated (see [11, 12, 13, 14]).

In this paper, we defined a Nörlund type matrix $A=(a_{nk})$. Then, we showed that $A=(a_{nk})$ is a linear and bounded operator on the sequence spaces $\ell_1$ and $bv$. Moreover, we calculated the fine spectrum and its subdivisions on the sequence spaces $\ell_1$ and $bv$.

As a natural continuation of this paper, the fine spectrum of the Nörlund type matrices on the Hahn sequence space $h$ and on the generalized Hahn sequence space $h_d$ and $h_d^p$, where $1<p<\infty$, which was defined and studied by Malkowsky at al [21], Malkowsky [22], and Tuğ at al [23], respectively, can be studied.

## 6. Author’s Contribution

We confirm that the manuscript has been read and approved by all named authors. We also confirm that each author has the same contribution to the paper. We further confirm that the order of authors listed in the manuscript has been approved by all authors.

## 7. Conflict of interest

There is no conflict of interest for this paper

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