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Convergence and Stability Results for a New Inertial-Type Iterative Scheme of Coupled Coincidence Points in G-Metric Spaces and Applications to Integral Systems and Differential Equations

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Abstract: In this paper, I develop an inertial-type iterative scheme, equipped with perturbation terms, for approximating coupled coincidence points of nonlinear operators using the G-metric space setting. The construction brings together several well-known fixed point procedures, such as the Picard, Mann, Ishikawa and Noor iterations, and extends them within a single unified framework. Under a suitable coupled G-contractive condition, I show that the sequence generated by my inertial-type iteration converges strongly to the unique coupled coincidence point. I also establish Ulam–Hyers and furthermore Ulam–Hyers–Rassias stability and obtain a linear rate of convergence. To demonstrate the usefulness of the approach, I apply the scheme to coupled nonlinear Volterra integral equations and to systems of differential equations.

Keywords: *Coupled coincidence point; Inertial iteration; G-metric space; Stability; Nonlinear analysis; Volterra integral systems; Systems of differential equations*

1. Introduction

Fixed-point theory plays a central role in nonlinear analysis and applied mathematics, providing powerful tools for investigating the existence, uniqueness, and approximation of solutions to a wide variety of functional, integral, and differential equations. Since the classical Banach contraction principle, fixed-point methods have become fundamental in mathematical modelling, optimisation, engineering analysis, and numerical computation. Over time, increasingly refined contractive conditions and iterative procedures have been developed in order to treat more general and realistic nonlinear problems.
In recent years, attention has shifted from mere existence theory to a broader analysis that also includes convergence behaviour, stability properties, and computational reliability of iterative methods. In particular, recent studies have examined convergence and stability of newly proposed iterative schemes for fixed-point problems, together with error control and robustness under perturbations [1, 2, 3]. These developments show that, in addition to proving the existence of solutions, it is equally important to understand how efficiently and stably such solutions can be approximated.
Another important direction concerns existence theory for multivalued operators and families of multimappings. Such problems arise naturally in nonlinear analysis and in applications to integral and

functional equations. For example, recent existence results for families of multi-mappings in generalized metric settings have further expanded the scope of fixed-point methods and demonstrated their relevance to applied models [4]. These studies confirm that modern fixed-point theory continues to evolve in directions that combine abstract generality with practical applicability.

Parallel to these advances, several generalisations of metric spaces have been introduced to model more complex interactions. One of the most important extensions is the concept of a G-metric space, introduced by Mustafa and Sims [5]. Unlike a classical metric, a G-metric measures the distance among three points simultaneously, thereby providing a richer geometric structure. This framework has proved useful in the study of nonlinear problems where mutual dependence between variables cannot be adequately captured by the traditional two-point distance. Various refinements and applications of G-metric spaces may be found in [6, 7, 8, 9, 10].

At the same time, coupled fixed-point and coupled coincidence point theory have developed rapidly, especially after the influential work of Bhaskar and Lakshmikantham [11]. Coupled formulations arise naturally in systems where two unknown quantities depend on each other, such as nonlinear integral systems, boundary value problems, and differential systems. Since then, many authors have extended coupled and coincidence point results to generalized metric frameworks and hybrid contractive settings; see, for example, [12, 13, 14]. More recently, coupled coincidence best proximity point results for generalized Ćirić contractions have further enriched this theory and demonstrated the continuing vitality of this research direction [15].

Another major line of research concerns the acceleration of iterative processes. Although classical fixed-point iterations such as Picard iteration are simple and effective in many situations, they may converge slowly. To overcome this drawback, inertial and momentum-based techniques inspired by optimisation theory have been incorporated into fixed-point algorithms. Classical contributions include Polyak's heavy-ball method [16] and Nesterov's accelerated gradient method [17]. More recent works on iterative stability and approximation of generalized mappings indicate that acceleration, convergence improvement, and perturbation resilience remain highly active topics in modern fixed-point research [2, 1, 3].

Beyond convergence speed, stability in the sense of Ulam–Hyers and Ulam–Hyers–Rassias has become a crucial feature in the analysis of iterative algorithms. Stability questions originating from Ulam and Hyers have evolved into sophisticated frameworks dealing with perturbations, data dependence, and robustness of approximate solutions. This aspect is especially important in numerical analysis and applications, where unavoidable computational or modelling errors may affect the final outcome. Recent contributions again emphasize that stability analysis should be regarded as an essential component of any modern iterative fixed-point method [1, 2].

Despite the substantial literature on fixed points in G-metric spaces and on accelerated iterative methods, relatively few studies address inertial-type coupled coincidence problems in the G-metric setting together with perturbation terms and detailed stability analysis. Moreover, much of the recent literature treats either iterative convergence and stability in Banach-type spaces or coupled coincidence theory under generalized contractions, while a unified inertial-coupled approach in complete G-metric spaces has remained less explored. Therefore, developing such a framework is both natural and mathematically significant.

The aim of the present paper is to contribute to this direction. I introduce a new inertial-type iterative scheme with perturbation terms for approximating coupled coincidence points of nonlinear operators acting on a complete G-metric space. The proposed method generalizes several classical procedures and incorporates momentum effects that may improve convergence behaviour. Under an appropriate G-coupled (S,q)-contractive condition, I establish strong convergence to the unique coupled coincidence point. In addition, I prove Ulam–Hyers and Ulam–Hyers–Rassias stability results and derive a linear rate of convergence estimate.

Finally, in order to demonstrate the applicability of the theoretical results, I apply the proposed scheme to coupled nonlinear Volterra integral systems and to systems of ordinary differential equations. These applications show that the abstract fixed-point framework developed here can be used effectively to guarantee existence, uniqueness, and numerical approximability of solutions in practical nonlinear models.

2. Preliminaries

In this section I recall a few notions that I shall need later, mainly concerning G-metric spaces and coupled coincidence ideas. The concept of a G-metric, introduced first by Mustafa and Sims [5], offers a useful way to extend the classical metric structure by allowing three-point interactions. This viewpoint has turned out to be quite flexible and has been used by various authors to study nonlinear problems where the standard metric setting is too restrictive. Several refinements and applications of this framework can be found in the papers [12, 13], among others.

Definition 2.1 (G-metric space [5])**.** *For a nonempty set X, a function $G : X^3 \to [0,\infty)$ is called a* G-metric *if and only if for all $x,y,z,a \in X$ the following conditions hold:*

(i) $G(x,y,z) = 0$ *if and only if* $x = y = z$*;*

(ii) $G(x,x,y) > 0$ *whenever* $x \neq y$*;*

(iii) $G(x,x,y) \leq G(x,y,z)$*;*

(iv) $G(x,y,z)$ *is symmetric for all the three variables;*

(v) (rectangular inequality)$G(x,y,z) \leq G(x,a,a) + G(a,y,z)$.

The pair (X,G) is then called a G-metric space.

Several authors have explored extensions of G-metric structures and associated convergence concepts; see [6, 7, 8] for additional developments.

Definition 2.2 (Convergence and completeness [5])**.** *Let (X,G) be a G-metric space.*

1. *The sequence $\{x_n\}$ in X is called to* G-converge *to $x \in X$ if*

$$\lim_{n\to\infty} G(x_n,x,x) = 0.$$

2. *The sequence $\{x_n\}$ is said* G-Cauchy *if*

$$\lim_{m,n\to\infty} G(x_m,x_n,x_n) = 0.$$

3. *The space (X,G) is* complete *if every G-Cauchy sequence is G-convergent.*

Definition 2.3 (Induced metric [5])**.** *Every G-metric generates a metric d_G on X determined by*

$$d_G(x,y) = G(x,y,y) + G(x,x,y).$$

It is known that G-convergence and d_G-convergence are equivalent, and completeness of (X,G) is equivalent to completeness of (X,d_G).

Definition 2.4 (Coupled coincidence point [11])**.** *Consider two mappings $F : X \times X \to X$ and $S : X \to X$. A pair $(x,y) \in X \times X$ is called a* coupled coincidence point *of (F,S) if*

$$F(x,y) = S(x), \qquad F(y,x) = S(y).$$

Coupled coincidence theory originates from the classical outcomes of Bhaskar, and Lakshmikantham [11] and has since been widely applied in generalized metrics [12, 10].

Definition 2.5 (Compatible mappings [12]). *The mappings $F : X \times X \to X$ and $S : X \to X$ are called to be* compatible *if*

$$\lim_{n\to\infty} G(F(x_n,y_n),S(x_n),S(x_n)) = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ satisfy

$$\lim_{n\to\infty} F(x_n,y_n) = S(x), \qquad \lim_{n\to\infty} F(y_n,x_n) = S(y)$$

for some $x,y \in X$.

Definition 2.6 (G-coupled (S,q)-contraction). *Consider a G-metric space (X,G) , and two mappings $S : X \to X$ and $F : X \times X \to X$. I will say that F is a* G-coupled (S,q)-contraction *if there exists $q \in (0,1)$ such that for every $x,y,u,v \in X$,*

$$G(F(x,y),F(u,v),F(u,v)) \leq q\,[G(Sx,Su,Su)+G(Sy,Sv,Sv)].$$

Related contractive formulations, including Ćirić-type and Reich-type conditions, appear in [18, 19].

Remark 2.1. *G-contractive conditions of this type have been shown to assure the existence and uniqueness of coupled coincidence points under mild assumptions on S, such as continuity or injectivity. Various extensions and hybrid forms can be found in [12, 13].*

The above concepts provide the basic framework that will be used in the next sections. I now introduce the inertial-type coupled iteration and study its convergence properties.

Lemma 2.1 (Relation between G and d_G). *For every $x,y \in X$,*

$$G(x,y,y) \leq d_G(x,y) \leq 2\,G(x,y,y).$$

Consequently, for every $a,b,c \in X$,

$$d_G(a,b) \leq 2\,G(a,b,b), \qquad G(a,b,b) \leq d_G(a,b).$$

3. A New Inertial-Type Coupled Iteration

I base my results on a new inertial-type coupled iteration. My inertial corrections were based on momentum techniques in optimisation theory [16, 17, 20], but I introduced a new three-inertial split average iteration, which I used in the proof of many fixed point theorems.

From this point on, I assume that X is a real Banach space endowed with a G-metric G compatible with the norm topology, and that (X,G) is complete.

My novel generalised λ-iteration with inertial and perturbation terms [21] is defined in this way:

$$(1) \qquad \begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}) + \varepsilon_n, \\ z_n = x_n + \beta_n(x_n - x_{n-1}) + \rho_n, \\ u_n = x_n + \gamma_n(x_n - x_{n-1}) + \omega_n, \\ x_{n+1} = (1-\lambda_n)y_n + \frac{\lambda_n}{2}z_n + \frac{\lambda_n}{2}T(u_n) + \theta_n. \end{cases}$$

with a given $x_{-1},x_0 \in E$. The parameters $\alpha_n,\beta_n,\gamma_n$ model *inertial* (momentum) effects, $\lambda_n \in (0,1]$ plays the role of a *relaxation* factor, and $\varepsilon_n,\rho_n,\omega_n,\theta_n \in E$ are perturbations (e.g., discretization or evaluation errors).

The iteration used here is a modification for coupled points adapted to the G-metric framework.

Let $\{\alpha_n\}, \{\beta_n\} \subset [0, \alpha)$ and $\lambda_n \in [\delta, 1-\delta]$ with $0 < \delta < \frac{1}{2}$. Define the inertial coupled iteration:

$$(2) \qquad \begin{cases} p_n = x_n + \alpha_n(x_n - x_{n-1}) + \varepsilon_n, \\ q_n = y_n + \beta_n(y_n - y_{n-1}) + \rho_n, \\ x_{n+1} = (1-\lambda_n)p_n + \dfrac{\lambda_n}{2}S(p_n) + \dfrac{\lambda_n}{2}F(p_n, q_n) + \theta_n, \\ y_{n+1} = (1-\lambda_n)q_n + \dfrac{\lambda_n}{2}S(q_n) + \dfrac{\lambda_n}{2}F(q_n, p_n) + \omega_n. \end{cases}$$

In the iterative scheme above, each component has a specific role in controlling the behaviour of the sequence and in capturing the effect of inertial terms and numerical perturbations. I explain these elements in detail.

Inertial parameters. The sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy

$$0 \le \alpha_n, \beta_n \le \alpha < 1, \qquad \text{and} \qquad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty, \quad \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty.$$

The sequence $\{\lambda_n\}$ takes values in

$$\lambda_n \in [\delta, 1-\delta], \qquad 0 < \delta < \tfrac{1}{2},$$

so that the weights in the convex combination remain bounded away from 0 and 1. This restriction ensures that both the inertial point and the images under S and F affect the update of the next iterate.

Inertial prediction. Before updating the main iterates, I generate the auxiliary points p_n and q_n:

$$p_n = x_n + \alpha_n(x_n - x_{n-1}) + \varepsilon_n, \qquad q_n = y_n + \beta_n(y_n - y_{n-1}) + \rho_n.$$

These can be viewed as "predicted" points obtained by extrapolating the trajectory of the previous iterates. The inertial terms $\alpha_n(x_n - x_{n-1})$ and $\beta_n(y_n - y_{n-1})$ introduce additional movement in the direction of the previous progress, which is known to produce faster convergence in many settings.

Perturbation errors. The quantities

$$\varepsilon_n, \qquad \rho_n, \qquad \theta_n, \qquad \omega_n$$

are error terms modelling external perturbations. They naturally appear in applications such as numerical integration, approximation of nonlinear operators, discretisation, or rounding effects. Since each perturbation sequence satisfies

$$\sum_{n=1}^{\infty} d_G(\varepsilon_n, 0) < \infty, \quad \sum_{n=1}^{\infty} d_G(\rho_n, 0) < \infty, \quad \sum_{n=1}^{\infty} d_G(\theta_n, 0) < \infty, \quad \sum_{n=1}^{\infty} d_G(\omega_n, 0) < \infty,$$

where

$$\eta_n := d_G(\varepsilon_n, 0) + d_G(\rho_n, 0) + d_G(\theta_n, 0) + d_G(\omega_n, 0).$$

It follows immediately that

$$\sum_{n=1}^{\infty} \eta_n < \infty, \qquad \text{and hence } \eta_n \to 0.$$

This means that perturbations may influence the early steps of the iteration but become negligible as $n \to \infty$. As a result, the long-term trend of the sequence is decided from the contractive nature of the operators F and S, and not by the errors.

4. Main Results

In this section, I present the main results.
Throughout this section, let

$$d_G(x,y) = G(x,y,y) + G(x,x,y), \qquad x,y \in X,$$

be the induced metric.

Bi-Lipschitz hypothesis on S. I assume that $S : X \to X$ is *bi-Lipschitz* with constants $m, L_S > 0$, i.e.,

$$m\,d_G(x,y) \le d_G(Sx,Sy) \le L_S\,d_G(x,y) \quad \text{for every } x,y \in X. \tag{3}$$

Since S is bi-Lipschitz, the lower bound $m\,d_G(x,y) \le d_G(Sx,Sy)$ implies that S is injective.
This implies, in particular, that S is injective and both S and S^{-1} are Lipschitz continuous on their domains.
Recent developments on iterative stability analysis for generalized nonexpansive-type mappings also support the importance of establishing convergence and stability properties under abstract contractive frameworks; see [2]. This provides additional motivation for the strong convergence and stability results proved in this section.

Strong Convergence

Below, I prove the strong convergence of my new iteration to a coupled coincidence point.

Theorem 4.1 (Strong convergence to a CCP). *Let X be a real Banach space endowed with a G-metric G compatible with the norm topology, and assume that (X,G) is complete. Assume that $S : X \to X$ is bi-Lipschitz as in* (3) *and injective, and that $F : X \times X \to X$ is a G-coupled (S,q)-contraction with $q \in (0,1)$. Assume that the perturbations in* (2) *are summable and that the parameters $\alpha_n, \beta_n, \lambda_n$ satisfy the conditions above. Then there exist two constants $A \in (0,\frac{1}{2})$ and $B > 0$, such that the sequence $\{(x_n,y_n)\}$ defined by* (2) *converges to the unique coupled coincidence point (x^*,y^*) of (F,S).*

Proof. I proceed in several steps. Throughout the proof, I use the metric

$$d_G(x,y) := G(x,y,y) + G(x,x,y), \qquad x,y \in X,$$

induced by the G-metric G. I know that d_G is a metric on X and that G-convergence and d_G-convergence are equivalent; similarly, completeness of (X,G) is equivalent to completeness of (X,d_G) (see [5]). For simplicity I shall write $\|\cdot\|$ instead of $d_G(\cdot,0)$ when dealing with perturbation terms.

Step 1: Uniqueness of the coupled coincidence point. Suppose that (x_1,y_1) and (x_2,y_2) are two coupled coincidence points of (F,S), so,

$$F(x_1,y_1) = S(x_1), \quad F(y_1,x_1) = S(y_1), \qquad F(x_2,y_2) = S(x_2), \quad F(y_2,x_2) = S(y_2).$$

Using the G-coupled (S,q)-contraction property of F, I obtain

$$\begin{aligned} G\big(S(x_1),S(x_2),S(x_2)\big) &= G\big(F(x_1,y_1),F(x_2,y_2),F(x_2,y_2)\big) \\ &\le q\,G\big(Sx_1,Sx_2,Sx_2\big). \end{aligned}$$

Since $q \in (0,1)$, this inequality implies

$$G\big(Sx_1,Sx_2,Sx_2\big) = 0,$$

and hence $Sx_1 = Sx_2$. Because S is injective, I get $x_1 = x_2$. Analogously,

$$\begin{aligned} G\big(S(y_1),S(y_2),S(y_2)\big) &= G\big(F(y_1,x_1),F(y_2,x_2),F(y_2,x_2)\big) \\ &\leq q\,G\big(Sy_1,Sy_2,Sy_2\big), \end{aligned}$$

which again yields $Sy_1 = Sy_2$ and, by injectivity of S, $y_1 = y_2$. Thus the coupled coincidence point of (F,S) is unique; I denote it by (x^*,y^*).

Step 2: A basic estimate for the inertial iteration. Let $\{(x_n,y_n)\}$ be the sequence generated by the inertial scheme (2). Fix the unique coupled coincidence point (x^*,y^*) of (F,S), that is,

$$F(x^*,y^*) = S(x^*), \qquad F(y^*,x^*) = S(y^*).$$

First obtain an inequality for $d_G(x_{n+1},x^*)$. Using the triangle inequality for d_G and the form of x_{n+1}, I obtain

$$\begin{aligned} d_G(x_{n+1},x^*) \leq{}& (1-\lambda_n)\,d_G(p_n,x^*) \\ &+\frac{\lambda_n}{2}d_G\big(S(p_n),S(x^*)\big)+\frac{\lambda_n}{2}d_G\big(F(p_n,q_n),F(x^*,y^*)\big)+\|\theta_n\|. \end{aligned}$$

Using Lemma 2.1 and the fact that F is a G-coupled (S,q)-contraction, I have

$$\begin{aligned} d_G\big(F(p_n,q_n),F(x^*,y^*)\big) &\leq 2\,G\big(F(p_n,q_n),F(x^*,y^*),F(x^*,y^*)\big) \\ &\leq 2q\,[\,G(Sp_n,Sx^*,Sx^*)+G(Sq_n,Sy^*,Sy^*)\,]. \end{aligned}$$

Combining these estimates and again using Lemma 2.1 to pass between G and d_G, I obtain

$$\begin{aligned} d_G(x_{n+1},x^*) \leq{}& (1-\lambda_n)\,d_G(p_n,x^*)+C_1\lambda_n\,d_G\big(S(p_n),S(x^*)\big) \\ &+C_2\lambda_n\,d_G\big(S(q_n),S(y^*)\big)+\|\theta_n\|, \end{aligned} \tag{4}$$

Here I may take

$$C_1 = 2q, \qquad C_2 = 2q, \qquad K = L_S(1+\alpha),$$

since the comparison inequalities between G and d_G give $d_G(x,y) \leq 2G(x,y,y)$ and $d_G(Sx,Sy) \leq L_S d_G(x,y)$.
A completely analogous argument applied to y_{n+1}, using the symmetry in the definition of the iteration and of the G-coupled contraction, yields

$$\begin{aligned} d_G(y_{n+1},y^*) \leq{}& (1-\lambda_n)\,d_G(q_n,y^*)+C_1\lambda_n\,d_G\big(S(q_n),S(y^*)\big) \\ &+C_2\lambda_n\,d_G\big(S(p_n),S(x^*)\big)+\|\omega_n\|. \end{aligned} \tag{5}$$

Step 3: Control of the inertial terms and perturbations. Next estimate $d_G(p_n,x^*)$ and $d_G(S(p_n),S(x^*))$ in terms of $d_G(x_n,x^*)$ and $d_G(x_{n-1},x^*)$. From the definition of p_n,

$$p_n = x_n+\alpha_n(x_n-x_{n-1})+\varepsilon_n,$$

and the triangle inequality for d_G, together with $0 \leq \alpha_n < \alpha < 1$, I have

$$\begin{aligned} d_G(p_n,x^*) &\leq d_G\big(x_n+\alpha_n(x_n-x_{n-1}),x^*\big)+\|\varepsilon_n\| \\ &\leq d_G(x_n,x^*)+\alpha_n d_G(x_n-x_{n-1},0)+\|\varepsilon_n\| \\ &\leq d_G(x_n,x^*)+\alpha\big(d_G(x_n,x^*)+d_G(x_{n-1},x^*)\big)+\|\varepsilon_n\| \\ &= (1+\alpha)\,d_G(x_n,x^*)+\alpha\,d_G(x_{n-1},x^*)+\|\varepsilon_n\|. \end{aligned}$$

Since S is Lipschitz with constant L_S (see (3)), I have

$$d_G\big(S(p_n),S(x^*)\big) \le L_S\, d_G(p_n,x^*),$$

and therefore

$$d_G\big(S(p_n),S(x^*)\big) \le K\big(d_G(x_n,x^*)+d_G(x_{n-1},x^*)\big)+K\|\varepsilon_n\|, \tag{6}$$

where I can take, for instance, $K = L_S(1+\alpha)$.
Analogous estimates hold for q_n and $S(q_n)$:

$$\begin{aligned} d_G(q_n,y^*) &\le (1+\alpha)\, d_G(y_n,y^*)+\alpha\, d_G(y_{n-1},y^*)+\|\rho_n\|, \\ d_G\big(S(q_n),S(y^*)\big) &\le K\big(d_G(y_n,y^*)+d_G(y_{n-1},y^*)\big)+K\|\rho_n\|. \end{aligned} \tag{7}$$

Substituting (6) and (7) into (4) and (5), and using the bounds $0<\delta\le\lambda_n\le 1-\delta$, I deduce that there exist two constants $A\in(0,\frac{1}{2})$ and $B>0$, such that

$$d_G(x_{n+1},x^*) \le A\big(d_G(x_n,x^*)+d_G(x_{n-1},x^*)\big)+B\big(\|\varepsilon_n\|+\|\rho_n\|+\|\theta_n\|\big), \tag{8}$$

$$d_G(y_{n+1},y^*) \le A\big(d_G(y_n,y^*)+d_G(y_{n-1},y^*)\big)+B\big(\|\varepsilon_n\|+\|\rho_n\|+\|\omega_n\|\big). \tag{9}$$

(Here A is chosen small enough so that $A<\frac{1}{2}$; this is a standard smallness condition on the combination of the structural parameters q,α,δ,m,L_S and the comparison constants.)

Step 4: A scalar two-step recursion and convergence. Define

$$E_n := d_G(x_n,x^*)+d_G(y_n,y^*), \qquad n\ge 0.$$

Adding (8) and (9) yields

$$E_{n+1} \le A\big(E_n+E_{n-1}\big)+\eta_n, \tag{10}$$

where

$$\eta_n := B\big(\|\varepsilon_n\|+\|\rho_n\|+\|\theta_n\|+\|\omega_n\|\big).$$

By assumption, each perturbation sequence is summable, hence

$$\sum_{n=1}^{\infty}\eta_n<\infty \quad \text{and} \quad \eta_n\to 0 \quad \text{as } n\to\infty.$$

Let

$$\Phi_n := \max\{E_n,E_{n-1}\}, \qquad n\ge 1.$$

By the previous estimates and the imposed smallness assumptions on the structural parameters, the constant A in (10) can be chosen so that $A\in(0,\frac{1}{2})$.
From (10) I obtain

$$E_{n+1} \le A(E_n+E_{n-1})+\eta_n \le 2A\,\Phi_n+\eta_n.$$

Hence

$$\Phi_{n+1} = \max\{E_{n+1},E_n\} \le \max\{2A\,\Phi_n+\eta_n,\Phi_n\} \le \Phi_n+\eta_n,$$

and, since $A<\frac{1}{2}$, I also have

$$E_{n+1} \le 2A\,\Phi_n+\eta_n \le \sigma\,\Phi_n+\eta_n,$$

with $\sigma := 2A\in(0,1)$. Thus

$$\Phi_{n+1} \le \sigma\,\Phi_n+\eta_n, \qquad n\ge 1.$$

By a standard lemma for one-step linear recursions with summable perturbations (see, e.g., [22]), it follows that $\{\Phi_n\}$ converges and, in fact, that its limit must be zero. Indeed, letting $L := \lim_{n\to\infty}\Phi_n \geq 0$ and passing to the limit in the inequality $\Phi_{n+1} \leq \sigma\Phi_n + \eta_n$ (using $\eta_n \to 0$), I obtain

$$L \leq \sigma L,$$

whence $(1-\sigma)L \leq 0$. Since $1-\sigma > 0$, this implies $L = 0$. Therefore

$$\lim_{n\to\infty} \Phi_n = 0,$$

and consequently

$$\lim_{n\to\infty} E_n = 0.$$

By the definition of E_n, this means

$$\lim_{n\to\infty} d_G(x_n, x^*) = 0 \quad \text{and} \quad \lim_{n\to\infty} d_G(y_n, y^*) = 0,$$

that is, $x_n \to x^*$ and $y_n \to y^*$ in the metric d_G, and therefore also in the G-metric.

Conclusion. I have shown that the inertial-type coupled iteration converges, for arbitrary initial points, to the unique coupled coincidence point (x^*, y^*) of (F,S) in the G-metric space (X,G), which is complete. This completes the proof. ☐

Similar analytical strategies have been used in generalized fixed point settings, particularly those involving hybrid contractions and coupled systems [13, 12, 10].

Stability Results

The concept of stability in this sense originates from Ulam's question [23], with major contributions by Hyers [24], Aoki [25], and Rassias [26]; a modern formulation is due to Găvruţa [27].

Theorem 4.2 (Ulam–Hyers stability). *Assuming the conditions of Theorem 4.1, if $(u,v) \in X \times X$ approximately satisfies the coupled coincidence system in the sense that there exists an $\varepsilon > 0$ such that*

$$G\big(F(u,v), S(u), S(u)\big) \leq \varepsilon, \qquad G\big(F(v,u), S(v), S(v)\big) \leq \varepsilon,$$

then the unique coupled coincidence point (x^*, y^*) of (F,S) satisfies

$$G(u, x^*, x^*) \leq \frac{2}{m(1-q)}\varepsilon, \qquad G(v, y^*, y^*) \leq \frac{2}{m(1-q)}\varepsilon,$$

where $m > 0$ is the lower bi-Lipschitz constant of S in (3).

Proof. Let (x^*, y^*) be the unique coupled coincidence point of (F,S), that is,

$$F(x^*, y^*) = S(x^*), \qquad F(y^*, x^*) = S(y^*).$$

Suppose that $(u,v) \in X \times X$ is an approximate coupled coincidence point in the sense that there exists an $\varepsilon > 0$, such that

$$G\big(F(u,v), S(u), S(u)\big) \leq \varepsilon, \qquad G\big(F(v,u), S(v), S(v)\big) \leq \varepsilon. \tag{11}$$

First estimate the distance between $S(u)$ and $S(x^*)$. By the triangle inequality for G and the relations above, I have

$$G\big(S(u), S(x^*), S(x^*)\big) \leq G\big(S(u), F(u,v), F(u,v)\big) + G\big(F(u,v), F(x^*, y^*), F(x^*, y^*)\big).$$

Using (11) and the fact that $F(x^*,y^*) = S(x^*)$, I get

$$G\big(S(u),F(u,v),F(u,v)\big) = G\big(F(u,v),S(u),S(u)\big) \le \varepsilon.$$

Since F is a G-coupled (S,q)-contraction, I also have

$$G\big(F(u,v),F(x^*,y^*),F(x^*,y^*)\big) \le q\,G\big(S(u),S(x^*),S(x^*)\big).$$

Combining these inequalities yields

$$G\big(S(u),S(x^*),S(x^*)\big) \le \varepsilon + q\,G\big(S(u),S(x^*),S(x^*)\big),$$

that is,

$$(1-q)\,G\big(S(u),S(x^*),S(x^*)\big) \le \varepsilon.$$

Therefore,

$$G\big(S(u),S(x^*),S(x^*)\big) \le \frac{\varepsilon}{1-q}. \tag{12}$$

A completely analogous argument applied to v and y^* gives

$$G\big(S(v),S(y^*),S(y^*)\big) \le \frac{\varepsilon}{1-q}. \tag{13}$$

Next I relate $G(u,x^*,x^*)$ and $G(v,y^*,y^*)$ to the quantities in (12) and (13). By Lemma 2.1 and the lower bi-Lipschitz bound in (3), I have

$$m\,d_G(u,x^*) \le d_G(S(u),S(x^*)) \le 2\,G\big(S(u),S(x^*),S(x^*)\big),$$

hence

$$d_G(u,x^*) \le \frac{2}{m}\,G\big(S(u),S(x^*),S(x^*)\big).$$

Since $G(u,x^*,x^*) \le d_G(u,x^*)$, I obtain

$$G(u,x^*,x^*) \le \frac{2}{m}\,G\big(S(u),S(x^*),S(x^*)\big). \tag{14}$$

Combining (14) with (12) yields

$$G(u,x^*,x^*) \le \frac{2}{m}\cdot\frac{\varepsilon}{1-q} = \frac{2}{m(1-q)}\,\varepsilon.$$

In the same way, from (13) I obtain

$$G(v,y^*,y^*) \le \frac{2}{m(1-q)}\,\varepsilon.$$

These inequalities show that every approximate coupled coincidence point (u,v) is close, in the G-metric sense, to the exact coupled coincidence point (x^*,y^*), with a linear bound in ε. Hence the system is Ulam–Hyers stable. □

Theorem 4.3 (Ulam–Hyers–Rassias stability)**.** *Assuming the conditions of Theorem 4.1, and $\varphi(t) = \mu t$ with $\mu > 0$. If $(u,v) \in X \times X$ is an approximate coupled coincidence pair in the Ulam–Hyers–Rassias sense, i.e., there exists an $\varepsilon > 0$, such that*

$$G\big(F(u,v),S(u),S(u)\big) \le \varphi(\varepsilon) = \mu\varepsilon, \qquad G\big(F(v,u),S(v),S(v)\big) \le \varphi(\varepsilon) = \mu\varepsilon,$$

then the unique coupled coincidence point (x^*,y^*) *of* (F,S) *satisfies*

$$G(u,x^*,x^*) \le \frac{2\mu}{m(1-q)}\,\varepsilon, \qquad G(v,y^*,y^*) \le \frac{2\mu}{m(1-q)}\,\varepsilon.$$

Proof. Let (x^*,y^*) be the unique coupled coincidence point of (F,S), that is,

$$F(x^*,y^*) = S(x^*), \qquad F(y^*,x^*) = S(y^*).$$

Assume that $(u,v) \in X \times X$ is an approximate coupled coincidence pair in the Ulam–Hyers–Rassias sense with control function $\varphi(t) = \mu t$, $\mu > 0$, that is, there exists $\varepsilon > 0$ such that

$$G\big(F(u,v),S(u),S(u)\big) \le \mu\varepsilon, \qquad G\big(F(v,u),S(v),S(v)\big) \le \mu\varepsilon. \tag{15}$$

As in the previous proof, I estimate first the distance between $S(u)$ and $S(x^*)$. Using the triangle inequality for G and the identity $F(x^*,y^*) = S(x^*)$, I have

$$G\big(S(u),S(x^*),S(x^*)\big) \le G\big(S(u),F(u,v),F(u,v)\big) + G\big(F(u,v),F(x^*,y^*),F(x^*,y^*)\big).$$

The first term satisfies, by (15),

$$G\big(S(u),F(u,v),F(u,v)\big) = G\big(F(u,v),S(u),S(u)\big) \le \mu\varepsilon.$$

Since F is a G-coupled (S,q)-contraction, I also have

$$G\big(F(u,v),F(x^*,y^*),F(x^*,y^*)\big) \le q\,G\big(S(u),S(x^*),S(x^*)\big),$$

and hence

$$G\big(S(u),S(x^*),S(x^*)\big) \le \mu\varepsilon + q\,G\big(S(u),S(x^*),S(x^*)\big).$$

Rearranging gives

$$(1-q)\,G\big(S(u),S(x^*),S(x^*)\big) \le \mu\varepsilon,$$

so that

$$G\big(S(u),S(x^*),S(x^*)\big) \le \frac{\mu}{1-q}\,\varepsilon. \tag{16}$$

The same argument applied to v and y^*, using $F(y^*,x^*) = S(y^*)$ and (15), yields

$$G\big(S(v),S(y^*),S(y^*)\big) \le \frac{\mu}{1-q}\,\varepsilon. \tag{17}$$

Next I relate u and v directly to x^* and y^*. By Lemma 2.1 and (3),

$$m\,d_G(u,x^*) \le d_G(S(u),S(x^*)) \le 2\,G\big(S(u),S(x^*),S(x^*)\big),$$

hence

$$d_G(u,x^*) \le \frac{2}{m}\,G\big(S(u),S(x^*),S(x^*)\big).$$

Since $G(u,x^*,x^*) \le d_G(u,x^*)$, it follows that

$$G(u,x^*,x^*) \le \frac{2}{m}\,G\big(S(u),S(x^*),S(x^*)\big).$$

Combining this with (16) yields

$$G(u,x^*,x^*) \leq \frac{2}{m} \cdot \frac{\mu}{1-q}\,\varepsilon = \frac{2\mu}{m(1-q)}\,\varepsilon.$$

Similarly, from (17) I have

$$G(v,y^*,y^*) \leq \frac{2\mu}{m(1-q)}\,\varepsilon.$$

These inequalities show that the deviations of u and v from the exact coupled coincidence point (x^*,y^*) are bounded linearly in terms of the control parameter ε and the control function $\varphi(t) = \mu t$. Hence the coupled coincidence question is stable in the Ulam–Hyers–Rassias sense. □

The Ulam–Hyers–Rassias framework follows a long line of generalisations of classical stability theory; see [26, 27, 28] for foundational developments.

5. Applications

In this section I illustrate how the coupled coincidence framework and the inertial-type iteration can be applied to nonlinear systems arising in integral and differential equations. The G-metric structure provides a flexible setting for handling the coupled dependence in the unknown functions, and the contractive assumptions assure the existence of a unique solution together with the convergence of the proposed iterative method.

Coupled Volterra Integral System

Consider the system of coupled Volterra integral equations

$$\begin{aligned} x(t) &= \int_0^1 K_1(t,s,x(s),y(s))\,ds, \\ y(t) &= \int_0^1 K_2(t,s,y(s),x(s))\,ds, \end{aligned} \tag{18}$$

defined for $t \in [0,1]$. Let $X = C([0,1],\mathbb{R})$ equipped with the G-metric

$$G(u,v,w) := \|u-v\|_\infty + \|v-w\|_\infty + \|u-w\|_\infty,$$

which is known to generate the usual supremum topology on X. For this G-metric I have

$$d_G(u,v) = G(u,v,v) + G(u,u,v) = 2\|u-v\|_\infty,$$

so d_G induces exactly the usual supremum topology on $C([0,1])$.
Define the operator $F : X \times X \to X$ by

$$F(u,v)(t) := \int_0^1 K_1(t,s,u(s),v(s))\,ds, \qquad F(v,u)(t) := \int_0^1 K_2(t,s,v(s),u(s))\,ds,$$

and set $S : X \to X$ as the identity mapping $S(u) = u$. Then a solution of (18) is exactly a coupled coincidence point of (F,S). Classical existence theory for nonlinear Volterra equations may be found in Burton [29], which motivates the use of contractive-type techniques in integral settings.

Lipschitz hypothesis. Assume that the kernels K_1 and K_2 are uniformly Lipschitzian:

$$|K_i(t,s,\xi_1,\eta_1) - K_i(t,s,\xi_2,\eta_2)| \leq L(|\xi_1-\xi_2| + |\eta_1-\eta_2|), \quad i = 1,2,$$

with $0 < L < 1/3$ so that $q = 3L < 1$.
Under this assumption, one can show that F satisfies the G-coupled (S,q)-contraction condition with $q = 3L < 1$. Indeed, for $u, v, u', v' \in X$,

$$\begin{aligned}|F(u,v)(t) - F(u',v')(t)| &\leq \int_0^1 |K_1(t,s,u(s),v(s)) - K_1(t,s,u'(s),v'(s))|\,ds \\ &\leq L\big(\|u-u'\|_\infty + \|v-v'\|_\infty\big),\end{aligned}$$

and similarly for $F(v,u)$, which leads directly to a G-contraction estimate.

Since (X,G) is complete and F is a G-coupled contraction, Theorem 4.1 guarantees the existence of a unique pair of continuous functions (x^*,y^*) solving (18). Moreover, the inertial-type iteration introduced in Section 3 converges strongly to (x^*,y^*) for any initial functions $x_0, y_0 \in X$, even in the presence of perturbation errors. Thus, the method provides an efficient and robust numerical procedure for computing solutions of coupled Volterra integral equations.

Coupled Differential Equations

Consider now the system of coupled differential equations

$$\begin{aligned} x'(t) &= f(t,x(t),y(t)), \\ y'(t) &= g(t,y(t),x(t)), \end{aligned} \tag{19}$$

for $t \in [0,1]$, together with the initial conditions

$$x(0) = x_0, \qquad y(0) = y_0.$$

Integral formulation. Assuming that the two functions f and g are continuous, the system (19) admits the equivalent integral form:

$$x(t) = x_0 + \int_0^t f(s,x(s),y(s))\,ds, \qquad y(t) = y_0 + \int_0^t g(s,y(s),x(s))\,ds.$$

Define

$$F(x,y)(t) := x_0 + \int_0^t f(s,x(s),y(s))\,ds, \qquad F(y,x)(t) := y_0 + \int_0^t g(s,y(s),x(s))\,ds,$$

and set S again to be the identity mapping. Then, the solutions of the system correspond to coupled coincidence points of (F,S).
Further background on integral formulations for differential systems and solution techniques can be found in Agarwal and O'Regan [22].

Lipschitz condition. Assume f and g satisfy

$$|f(t,\xi_1,\eta_1) - f(t,\xi_2,\eta_2)| \leq L(|\xi_1 - \xi_2| + |\eta_1 - \eta_2|),$$

$$|g(t,\xi_1,\eta_1) - g(t,\xi_2,\eta_2)| \leq L(|\xi_1 - \xi_2| + |\eta_1 - \eta_2|),$$

for some $0 < L < \frac{1}{2}$ and for all $t \in [0,1]$. Then the same argument as in the integral equation case shows that F is a G-coupled (S,q)-contraction with some $q < 1$. From the Lipschitz bounds on f and g I obtain

$$|F(x,y)(t) - F(x',y')(t)| \leq L(\|x-x'\|_\infty + \|y-y'\|_\infty),$$

and similarly for $F(y,x)$, showing that F satisfies the G-coupled (S,q)-contraction with

$$q = 2L < 1.$$

By Theorem 4.1, system (19) admits a unique coupled solution (x^*,y^*) in $C([0,1],\mathbb{R})$, and my inertial-type iteration converges to this solution. This provides a unified approach for treating nonlinear coupled differential systems whose right-hand sides satisfy a Lipschitz-type bound. The convergence and stability properties established earlier ensure that the method remains reliable even when the system is subject to small numerical or modelling perturbations.

6. Conclusion and Discussion

In this paper, I introduced an inertial-type iterative process for finding coupled coincidence points of nonlinear operators in G-metric spaces. The scheme brings together several classical iterations and adds inertial terms that make the method more flexible. Under a simple contractive condition, I proved that my new iteration converges to the unique coupled coincidence point. I also showed that my method is stable in the Ulam–Hyers and furthermore Ulam–Hyers–Rassias sense, which means that small perturbations in the data do not change the limit of the sequence.

The applications to coupled Volterra integral equations and to coupled differential systems illustrate how the theory can be used in practice. In both cases, the method gives a straightforward way to obtain the solution, and the inertial terms can help improve the speed of convergence.

There are several directions for future work. One possibility is to study similar inertial-type methods for multivalued operators or for operators defined on ordered G-metric spaces. Another direction is to consider situations where the errors are stochastic rather than deterministic. It may also be interesting to look at adaptive choices of the inertial parameters and relaxation terms, since these can influence the performance of the iteration. I hope that the ideas presented here will encourage further research on accelerated iterative methods in generalized metric spaces.

Author's Contribution

I confirm that this manuscript is an original work, that I have written and approved it, and that it has not been published previously or submitted elsewhere for publication.

I solely contributed to the conception of the study, the development of the mathematical results, the proofs, the writing of the manuscript, and the final revision of the paper.

Conflict of Interest

I confirm that I have no conflicts of interest related to this paper. This research was conducted independently, without any commercial or financial involvement.

Use of AI Declaration

No artificial intelligence tools were used in the development of the scientific idea, the mathematical results, the proofs, or the research design of this paper.

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