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# Recent Advances in Approximate Methods for Predicting Nonlinear Vibrations in Cantilever Beams and Plates: A Review

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**Abstract:** This study provides a comprehensive overview of the significance of studying the nonlinear vibration characteristics of beam-like structures, such as cantilever pillars and plates, that have non-linear vibration characteristics that deserve special attention. Furthermore, beam-like members such as radio wires, rotor edges, airplane wings, supersonic airfoils of high rises, and others are used in building construction works, and they are engineered strategically to withstand bowing sideways. However, dynamic analysis when these structures is highly subjected to alternating and large axial strains, complex and often nonlinear, and these analyses may demand advanced modelling and analytical techniques which do not exist. The dynamic behavior of flexible structures is described using a set of equations that includes non-linear ordinary differential equations and such equations are tackled by the Ritz-Galerkin approach which will be covered in this paper, in comparison with the Galerkin and Lindstedt-Poincaré techniques, it demonstrates higher accuracy and lower cost, thus providing essential additional information on how to construct models of nonlinear vibrational systems properly. This study has addressed the limitations of the applicability of linear beam theory and pointed out the importance of nonlinearities for the dynamic behavior of beam-like structures. It discusses various types of nonlinearities that significantly affect the beam model motion equations, which are extremely useful for engineers and scientists. The research concentrated on the utilization of approximation techniques, namely the Galerkin method and the Lindstedt-Poincaré approach, in the analysis of beam vibration issues characterized by nonlinearity. This thoroughly examines the problem of nonlinear vibrations in cantilever beams and plates. It examined recent efforts in the advancement of approximation approaches for forecasting and assessing the nonlinear dynamic behavior of structural components.

**Keywords:** *Nonlinear Vibration; Cantilever Beam; Approximate Methods; Galerkin Method; Lindstedt-Poincaré Method.*

## 1. Introduction

A beam is usually slender and is designed to resist lateral action through the bending of the member; it forms a basic part of most structures, like antennas, helicopter rotor blades, airplane wings, tall buildings, and towers. Operations dealing with the structural integrity of the above beam-like structures have a significant magnitude of importance for engineers, as they are vulnerable to frequent exposure to dynamic loads. One of the most common theories used for small displacement analysis is the linear beam theory. This theory allows the calculation of the natural frequencies, mode shapes, and responses due to excitations. However, when the displacements become of great value, the traditional linear beam theory does not so accurate a prediction of the dynamic characteristics of the system. The appearance of geometric and other nonlinearities is only important when highly flexible beams develop substantial displacements. For the previously independent vibration modes, the presence of nonlinearities may interlink them: the modes can interact and exchange energy [1-3] . The modelling of beam elements, especially when large deflections are taken into consideration, nonlinearities such as nonlinear inertia, curvature, shear deformation, Poisson effects, and warping must be considered. Apart from geometric and inertial nonlinearities, beam-like structures dynamic behavior can be much influenced by damping processes. Cantilevers can undergo large oscillations of large amplitudes for excitations at the base of very small magnitudes. However, it is rather difficult to predict the behavior of a cantilever undergoing oscillations of very large amplitudes with sufficient accuracy due to the presence of several forms of nonlinearities, geometric nonlinearities as well as inertial nonlinearities arising from centerline inextensibility [4]. Nonlinear beam vibration with time-dependent boundary conditions means the deflection of the beam borders, which are constrained to undergo time-varying displacements. Such an incidence can be found in complex arrangements that involve one structural member interacting with another or cases where the beam is subjected to time-varying external stresses. The beam's vibration is nonlinear since large amplitudes have been developed, which creates huge changes in the beam deflection and responsiveness [5]. There are three primary methods for analyzing nonlinear vibrations: Lindstedt's perturbation method, the iterative method, and the Ritz-Galerkin method. While significant progress has been made in nonlinear vibration analysis for beam structures, further research is needed to improve the accuracy and efficiency of both modelling and analytical techniques. Recent advancements in approximation methods for predicting nonlinear vibrations in cantilever beams and plates have been scarcely explored in detail. This review aims to emphasize the importance of studying nonlinear vibrations in cantilever beams and plates, focusing on the limitations of linear beam theory. It highlights the necessity of nonlinearities in obtaining realistic dynamic response prediction of structures, determines the most significant nonlinearities for beam modelling and analysis, and outlines current developments in approximate methodologies.The review concludes with helpful recommendations to researchers and engineers, guiding the development of more effective and accurate modelling and analysis methodologies for beam-like structures. The paper aims at a comparison of the nonlinear vibration behavior of beam structures, with a special emphasis on Lindstedt-Poincaré and Galerkin methods, analytical and numerical solutions, and examining a combined approach to enhancing the precision of the solution and computational efficiency for nonlinear vibration problems.

## 2. Lindstedt-Poincaré Method

Mathematicians typically use perturbation as a technique to approximate complex problems that are impossible to solve exactly [6]. The main methods of solving nonlinear differential equations refer to the perturbation methods. Among them, the so-called Lindstedt-Poincaré method is rather general and powerful. It applies to nonlinear oscillators for which the restoring force is not linear. Recent enhancements to the Lindstedt-Poincaré method diminish processing expenses while maintaining accuracy. A streamlined method proposed by a study [7] is effective for higher-order nonlinear free vibrations. This approach facilitates quicker analysis of systems with several vibration modes and simplifies the resolution of nonlinear differential equations. The streamlined method aligns effectively with numerical solutions in systems exhibiting significant nonlinearities. This approach may assist engineers and academics in precisely and efficiently analyzing complex vibrating systems. The suggested approach reduces ineffective higher-order expansions, boosting convergence while maintaining solution accuracy. Although their work mostly concentrates on free vibrations, a similar simplification can help forced vibrations in elastic beams. The governing equation for the nonlinear free vibration of a beam using the SLP method:

$$\ddot{x} + \omega^2 x + \epsilon f(x, \dot{x}) = 0 \quad (1)$$

Simplified perturbation method (SLP) in Equation (1), $\epsilon$ is the small nonlinearity parameter. Eliminating pointless higher-order expansions in the Lindstedt-Poincaré method greatly reduced computational complexity without sacrificing accuracy, hence simplifying the procedure [1]. Particularly for mild to moderate nonlinearities, their results revealed that this method produces very good approximative frequency correction in line with numerical simulations. Moreover, by displaying a faster convergence rate than traditional perturbation methods, their method assured consistent findings with fewer processing steps. Especially for higher-order approximations, the method proved to be rather beneficial since it is over 40% more efficient than conventional Lindstedt-Poincaré methods. The work presented a useful method to find frequency adjustments with much fewer terms by using an effective power-series expansion, hence balancing accuracy with computing practicality. A method is a strong candidate for merging with Galerkin-based beam solutions since the technique effectively approximates higher-order corrections while decreasing the number of iterations required. The ability of the simplified Lindstedt-Poincaré approach to effectively determine frequency changes while decreasing computational complexity is its main advantage. In many nonlinear vibration problems, though, a more methodical technique is needed to convert the governing equations into a form allowing an iterative solution. The introduction of a new variable $\alpha = x(t)$, allows the differential equation to be recast in a perturbation framework. This change enables the expression of the problem in terms of a tiny parameter $\alpha$, which denotes the amplitude of oscillation, and generates a power-series expansion method for nonlinear beam problems [8-10].

There are two effective tools, such as the Runge-Kutta (RK4) method, along with the Multiple-Scales (MS) and Multiple-Scales Lindstedt-Poincaré (MSLP) methods, for the analysis of nonlinear systems. These approaches allow for more precise solutions, especially when linear models fail to capture the complex nonlinear behaviour of the system. When looking at nonlinear systems in this way, numerical models using the Runge-Kutta (RK4) method, the Multiple-Scales (MS) approach, and the Multiple-Scales Lindstedt-Poincaré (MSLP) strategy work well. These methods work well to fix the cantilever beam system's natural complex nonlinearities, which helps us learn more about how it works. Since nonlinear activities like frequency shifts and mode coupling define the actual system behavior, linear models cannot fully depict the situation. Particularly in such a system as a suspension beam with a breathing crack [3], such effects can have dramatic impacts on behavior. Advanced techniques need to be used with nonlinearities included if one is going to be able to truly forecast this behavior. Optimal solutions are able to be derived from mathematical techniques such as Multiple-Scales (MS) and Multiple-Scales Lindstedt-Poincaré (MSLP) methods, and also numerical techniques such as the Runge-Kutta (RK4) method. These techniques provide us with better estimations, particularly in scenarios where linear models fail, which accounts for the system's complex nonlinear behavior.

A. Multiple-Scales (MS) Method: Assume an asymptotic expansion of the displacement $q(t)$ as:

(2) $$q(t) = q_0(t, T_0, T_1) + \epsilon q_1 1(t, T_0, T_1) + O(\epsilon^2)$$

while $T_0 = t$ and $T_1 = \epsilon t$, at $O(\epsilon)$, we obtain the linear response equation, and at $O(\epsilon_2)$ the nonlinear terms are handled by eliminating secular terms. The Multiple-Scales (MS) technique divides the solution into more straightforward components to approximate the simulation of nonlinear vibration system responses, i.e., a cantilever beam. It approximates nonlinearity by stretching the displacement $q(t)$ in powers of a small parameter ϵ. It enables you to avoid unphysical growth in solutions, decouple the slow and fast time scales of the system, and manage nonlinear effects at different orders. Finally, the approach yields more accurate approximations to complicated nonlinear systems, where the conventional techniques can fail.

Multiple-Scales Lindstedt-Poincaré (MSLP) Method: For MSLP, introduce a frequency correction

(3) $$\Omega = \omega_n + \epsilon\Omega 1 + O(\epsilon^2)$$

This adjustment improves the accuracy of the approximation for nonlinearities. With a frequency correction application, the Multiple-Scales Lindstedt-Poincaré (MSLP) approach has to increase in the accuracy of solutions for nonlinear systems. This adjustment provides more precise approximations where other methods may not be able to reach with alterations in the natural frequency from nonlinearity [8,11].

B. Runge-Kutta (RK4) Method: The system is also solved numerically using the fourth-order Runge-Kutta method:

$$y_{n+1} = y_n + \frac{h}{6}\,(k_1 + 2k_2 + 2k_3 + k_4) \quad (4)$$

where the intermediate values $k_i$ are computed based on the function evaluations. The Runge-Kutta (RK4) method intends to provide differential equations with a precise numerical solution, particularly in cases when exact analytical solutions are either difficult or impossible to find. Using intermediate values $k_1$ helps solve the nonlinear differential equations controlling the motion of the cantilever beam by iteratively approximating the solution over small time steps. $k_2$ helps improve the accuracy of the solution at each step through intermediate values. This approach is extensively applied because of its harmony between computing economy and precision [12].

### 2.1. Nonlinear Behavior in Elastic Beams

One of the most extensively used structural members is the elastic beam. Long before the elasticity theory was developed, scholars had already carried out extensive studies on elastic beams. It's hard to solve the nonlinear flexural equation of elastic beams [13] without taking into account the properties of the material, its strength and its constitutive relationships. Over the past decades, the majority of research in nonlinear vibration has focused on the application of approximate methods. A multidimensional Lindstedt–Poincaré (MDLP) method was used by [11] as a rough way to solve the nonlinearity in the beam's governing equation when it moves in one direction. In this case, expanding the governing equation answer into harmonics and perturbing to achieve near answers yields the solution. This allowed them to analyze the forced response of a beam going in one direction with an internal resonance between the first two transverse modes. MDLP response curves have the same internal resonance as thin plates without beams. This is because all of these systems are cubic nonlinear and have a similar frequency distribution. We compared the results from the MDLP method and the incremental harmonic balance method and found that the MDLP method is easier to use and more effective for studying the nonlinear vibration of systems that move in one direction compared to other perturbation methods for single-degree-of-freedom system analyses [11].

### 2.2. Nonlinear Vibration of a cantilever beam with point load and lumped mass

The Lindstedt-Poincaré approach is explained in detail, along with recent advancements and practical uses. The method's ability to solve nonlinear differential equations, particularly those with nonlinear restoring forces, is thoroughly justified at the outset of the argument. Recent advancements, such as the simplified technique in [7], highlight the method's effectiveness and soundness for higher-order nonlinear systems. After that, the method shifts to particular uses like MDLP for internal resonance systems [11] and cantilever beams with lumped masses [9]. These circumstances illustrate that the approach is flexible and superior to traditional perturbation techniques. Figures such as the frequency response curves in [6] and the energy harvester model in [10] help to clear the uncertainty and show the effectiveness of the technique. Demonstrating its adaptability, the technology is used in microbeam analysis and energy collection [14]. Talks on spinning machines [13] and hard-coating laminate plates [15] stress the adaptability and precision of the technique.

Combining the latest developments, like the simpler techniques in [4,7], this study demonstrates the ongoing evolution and application of the method in contemporary engineering. The Lindstedt-Poincaré method has proven to be highly adaptable in solving a variety of technical problems, especially in understanding complex vibrational processes. In a study [16], nonlinear damping devices were used to regulate high-frequency rotational oscillations in self-excited drill string vibrations. The comparison between analytical solutions and time-domain simulations revealed the method's efficacy in predicting torsional dynamics. The findings also highlighted how nonlinear stiffness improves damping efficiency and contributes to overall system stability. In 2017, the study [9] used the multiple-scales Lindstedt-Poincaré (MSLP) method to look into the forced vibration of a cantilever beam with a lumping mass as shown in Figure 1. This figure illustrates the energy harvester model of a cantilever beam, elucidating the impact of external forces on the system. The apparatus comprises a lumped mass positioned at the free end of the beam. This mass influences the beam frequency. This model offers substantial insights into the nonlinear vibrational behavior of cantilever beams under forced excitation. This is very accurate in comparison to alternative approximation approaches. The equation of motion and boundary condition of the beam are presented using the variational approach based on the extended Hamiltonian principle.

$$m\ddot{y} + M_t\delta(s-L)\ddot{y}(L,t) + c_y\dot{y} + EIy^{iv} = F_A\cos(\Omega t)\,\delta(s-L) - EI[y'(y'y'')']' - \frac{1}{2}\left\{y'\int_L^s\left[\int_0^s y'^2 ds\right]''ds\right\}' + mg[(s-L)y''+y'] + mg\left[(s-L)\frac{3y'^2y''}{2} + \frac{y'^3}{2}\right] \tag{5}$$

It involves significant terms such as inertia, damping, and flexural rigidity in this dynamic equation, considering the nonlinear vibration of the cantilever beam due to harmonic excitations from the external world. It simplifies the equation into a system of coupled nonlinear ordinary differential equations that can be readily solved with the Multiple-Scales (MS) and Multiple-Scales Lindstedt-Poincaré (MSLP) methods. These techniques prolong the validity of analytical approximations, especially in cases having strong nonlinearities where normal perturbation methods fail. The system is subsequently analyzed using the MS and MSLP approaches [9,13] after further reduction to a system of coupled nonlinear ordinary differential equations. Particularly in cases of strong nonlinearity when normal perturbation methods fail, these approaches prolong the validity of analytical approximations. These parameters make it easier to model the actual response of the beam to large deformations. The equation below is the improved formulation for the nonlinear vibration of the beam. The tip mass cantilever beam's equation of motion due to damping and flexural stiffness is below: Equation of motion for nonlinear vibration of the beam:

$$m\ddot{y} + M_t\delta(s-L)\ddot{y}(L,t) + c\dot{y} + EIy^{(iv)}(s,t) = FAcos(\Omega t)\delta(s-L) - EIy(s,t)\frac{dy}{ds} - \int_0^L y(s,t)\left(\frac{dy}{ds}\right)^2 ds \tag{6}$$

The parameters and notations follow the conventions in previous research [9,15,17], where: $m$ is the mass per unit length of the beam, $M_t$ is the lumped mass at the free end of the cantilever, $c$ is the coefficient of linear viscous damping per unit length, $E$ is Young's modulus, $I$ is the beam's cross-sectional moment of inertia,g is the gravitational acceleration represents the arclength along the beam. $t$ is time, $\delta(s-L)$ is the Dirac delta function at the free end, $\Omega$ is the excitation frequency, $FA$ is the amplitude of the external force, and $y(s,t)$ is the transverse displacement of the beam as a function of position $s$ and time $t$. The nonlinearities, including frequency shifts, mode coupling and asymmetric frequency responses, have a considerable impact on the system's behavior. For accurate predictions, the inclusion of these nonlinear effects is essential. As shown in prior studies, particularly the investigation of a cantilever beam with a breathing crack [3,18] accurate forecasts require more advanced analytical techniques, as these nonlinearities significantly alter the system's behavior. From the results, it is obvious that with the MSLP method, the frequency response curve of the first mode is in excellent agreement with that obtained by the fourth-order Runge-Kutta method within the concerned beam deflection. However, the frequency response curve of the first mode obtained by the multiple-scales method deviates when the beam deflection is large, as shown in Figure 2.

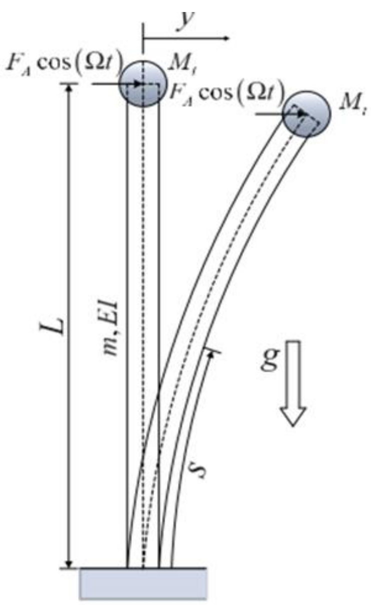


Figure 1: The scheme of the cantilever beam energy harvester with mass on the free ends.

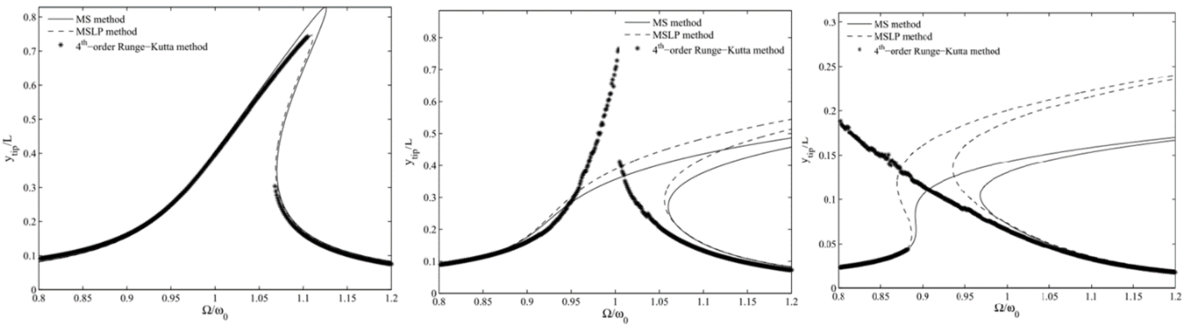


Figure 2: Comparison of the first, second, and third mode Frequency Response Curves (FRCs) using the MS method, MSLP method, and numerical simulation [1].

The frequency response is also hardly predictable by either the multiple-scales method or the MSLP method when the excitation frequency is around either the second or third natural frequency. The study provides insight into the effectiveness of these methods for analyzing the forced vibration of cantilever beams with lumped mass [9]. Further advancements, such as [7], make the Lindstedt-Poincaré technique easier to manage forced vibration difficulties in nonlinear systems while keeping its validity. These improvements improve approach performance by simplifying higher-order nonlinearity calculations without losing accuracy. The forced vibrations of the damped duffing oscillator examined at [6] as:

(7) $$\ddot{u} + \omega_0^2 u + 2\varepsilon^2 \mu \dot{u} + \varepsilon \alpha u^3 = \varepsilon^2 \mathrm{f} cos \Omega t$$

After examining a higher order of perturbation method, to re-order the excitation and damping, conduct the fast and slow time scales as $T_0 = t, T_1 = \varepsilon t\ and\ T_2 = \varepsilon^2 t$. Then they examined an MSLP method as

(8) $$\omega^2 u'' + \omega_0^2 u + 2\varepsilon^2 \mu \omega u' + \varepsilon \alpha u^3 = \varepsilon^2 f cos \frac{\Omega}{\omega} T_0$$

Again, used a fast and slow time scale as cap T sub 0 equals tau, cap T sub 1 equals script epsilon tau, andcap T sub 2 equals script epsilon squared, tau, used a fast and slow time scle as $T_0 = \tau, T_1 = \varepsilon\tau\ and\ T_2 = \varepsilon^2 \tau$ where time transformation $(\tau) = \Omega t$. MSLP shows non-linear interactions, and its rhythmic strength changes depending on the input frequency. The findings indicate that the first mode frequency response aligns with the advanced expectations of the fourth-order Runge-Kutta method. Nevertheless, increased deflections result in inconsistencies.

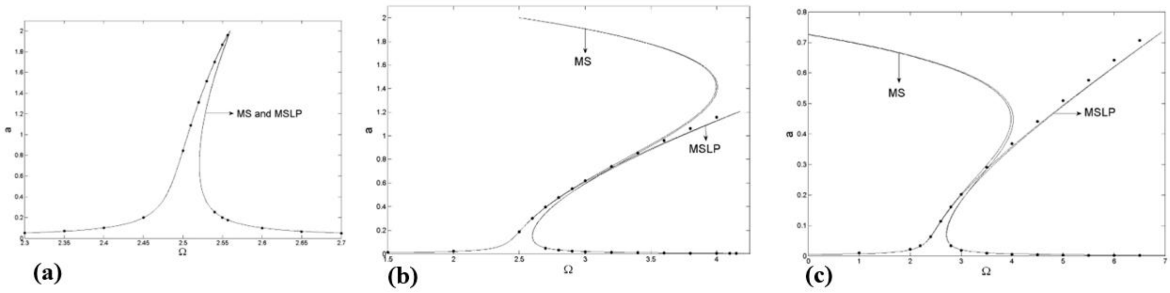


Figure 3: Frequency Response Curve Comparison for MS, MSLP Methods, corresponding to (a) α=1, (b) α=100 (c) α=1000 [2]

The methods compared in the Figure 3 MSLP data illustrating how the cantilever beam energy collector reacts to strong shocks. The added weight at the free end of the beam changes how it vibrates. It depicts the impact of fracture depth on resonance frequencies and amplitudes using frequency response graphs. These results are the same as the ones from the nonlinear dynamic equation explained earlier. This shows how important it is to include nonlinearities in predictions about structures, as beams do not always move in a perfectly straight line; their motion can be influenced by nonlinearities, such as cracks, varying stiffness, or external forces. Understanding these effects is crucial in predicting structural responses accurately. Based on nonlinear factors like lumped mass or fracture depth, the method can accurately predict how a structure will behave, as shown by the frequency response curves from [3, 9]. This demonstrates the practical application of advanced nonlinear methods to predict and analyze actual actions. They developed and improved nonlinear dampening techniques to suppress drill string vibration self-excited oscillations with the Lindstedt-Poincaré method [16]. Time-domain simulations verified the validity of torsional vibration prediction by the method. According to the study, the Lindstedt-Poincaré method works better to model situations with large changes in shape, changes in the resonance frequency, and energy transfer between vibrational modes. This makes it a powerful tool in studying complicated nonlinear dynamics. Aside from such situations, the method has worked well in other technical uses, important mention being in solving complex vibrational problems. For example, [16] used it to design nonlinear damping devices to control high-frequency torsional oscillations in self-excited drill string vibrations. Comparing analytical results with time-domain simulations confirmed the method's torsional dynamics correctness.

Prior research [9] indicates that the MSLP methodology is both efficient and effective for analyzing such systems, even in the presence of significant deformations. Figure 2 investigated the frequency response curves for the Multiple Scales (MS) method, the Multiple Scales Lindstedt-Poincaré (MSLP) method, and the computer models using the same set of parameters. Particularly at higher frequencies, the MSLP method provides an exact prediction of the system's behavior, which rather roughly corresponds with computer simulations. In contrast, the MS method shows some discrepancies from numerical data and suggests that under these specific conditions, the MSLP method is more appropriate for the study of the system's response. The narrative demonstrates the outcomes of the MSLP method, which admirably illustrates the nonlinear interaction modelling and study. The study [3] gives a good look at a beam with a breathing crack, showing how nonlinearities greatly impact the system's response. The dynamic equation of motion of the cracked beam was governed as:

$$M\ddot{U} + C\dot{U} + (K_{001} - K_t)U = F \tag{9}$$

Where the stiffness matrix $K_t$ is time-variant due to the repetitive crack opening and closing. The nonlinear equation derived as

(10) $$M_C\ddot{\eta}_C + C_C\dot{\eta}_C + K_C\eta_C - K_T\eta_C \sum_{n=1}^{N} k_n \cos(n\omega t + \varphi_n) = F_C$$

Where $M_C = \psi_C^T M \psi_C, C_C = \psi_C^T C \psi_C, K_C = \psi_C^T K_{001} \psi_C, K_T = \psi_C^T K_{02} \psi_{C,} F_C = \psi_C^T F, \psi_C \eta_C$ and

$\psi$ is the matrix of the modal transformation.

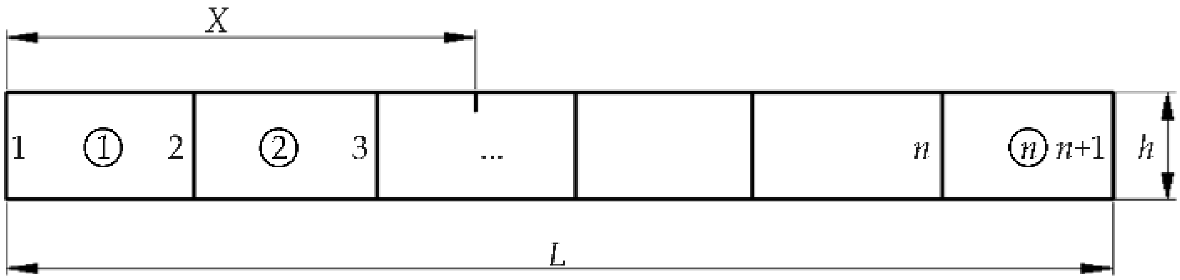


Figure 4: Discretisation of the beam with the breathing crack [3]

Figure 4 illustrates a cantilever beam exhibiting a breathing fracture, demonstrating how a singular location can compromise structural integrity. It hinders movement and responsiveness. The discretised method, which decomposes the beam into components, accurately captures these effects and delivers precise numerical analysis. The distribution of stiffness post-fracture influences energy dissipation and frequency response. Fracture models are essential for comprehending how structures react to varying stresses. Furthermore, as shown in [19], the approach has shown success in simplifying challenging beam vibration issues, lowering computational requirements while preserving great precision. The objective is to diminish the disparity between anticipated and actual responses. The balance of the residual or error over the affected region is attained using weighing methods. The authors used the Duffing equation as two methods for evaluating Lindstedt–Poincaré–type perturbation methods for the nonlinearity, the governing equations.

(11) $$\ddot{x} + \omega_o^2 x + \varepsilon^2 x^3 = 0,$$

where its solution is;

$$x(t) = x_o(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots$$

and fundamental frequency

$$\omega^2 = \omega_o^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots\ldots$$

after removing the secular terms and substituting the $\tau = \omega t, d/dt = \omega d/dt$ into equation

$$\omega^2 x'' + \omega_o^2 x + \varepsilon x^3 = 0$$

the first approximate solution was considered as:

(12) $$(\omega_o^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2)(x_0'' + \varepsilon x_1'' + \varepsilon^2 x_2'') + \omega_o^2(x_o + \varepsilon x_1 + \varepsilon^2 x_2) + \varepsilon(x_o + \varepsilon x_1 + \varepsilon^2 x_2)^3 = 0$$

The second approximate solution was considered as:

(13) $$\omega^2 (x_0'' + \varepsilon x_1'' + \varepsilon^2 x_2'') + (\omega^2 - \varepsilon\omega_1 - \varepsilon^2\omega_2)(x_o + \varepsilon x_1 + \varepsilon^2 x_2) + \varepsilon(x_o + \varepsilon x_1 + \varepsilon^2 x_2)^3 = 0$$

Both method calculated to find the second response and frequensies, their results showed in Figure 5, showed that the both methods are concide with the exact solution. They concluded that both methods have no significant effect on over another but first method is not valid for larg parameters.

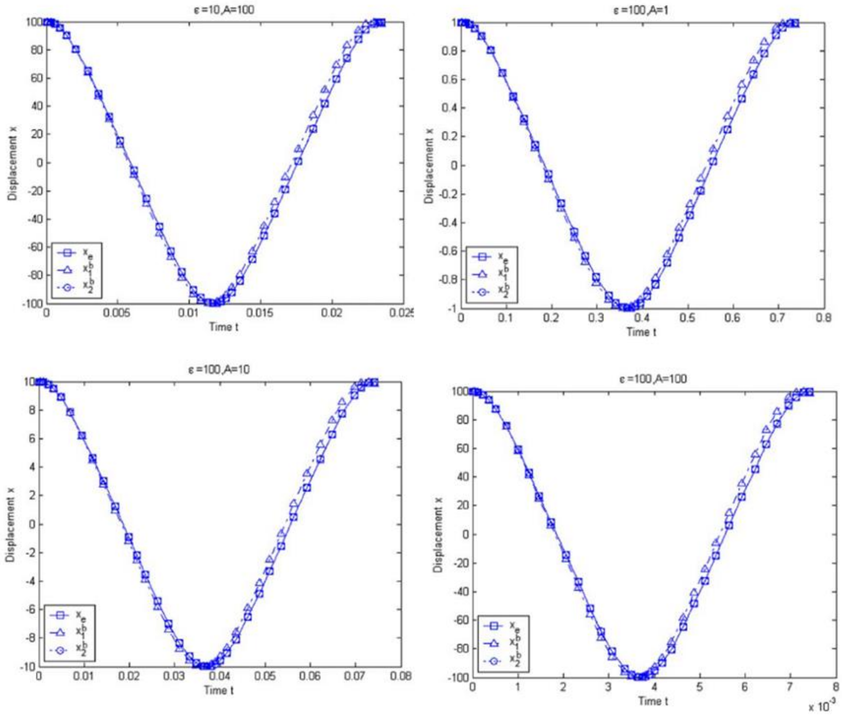


Figure 5: Comparison between approximate solutions with the exact solution [19]

The dynamic equation results from the studies [9, 15, 17] show that frequency response curves often show the difference between linear and nonlinear systems. Nonlinear effects cause resonance shifts, asymmetric frequency responses, and higher amplitudes at certain frequencies. For the mode shapes and energy exchange, linear models do not consider nonlinearities, which result in mode coupling and energy transfer between modes. Studies reveal that nonlinear methods, including Lindstedt-Poincaré and Galerkin, are more accurate than linear methods at predicting behaviors in the actual world. To learn more about how beams behave when they are not moving in a straight line, gives a good look at a beam with a breathing crack, showing how nonlinearities greatly impact the system's response.

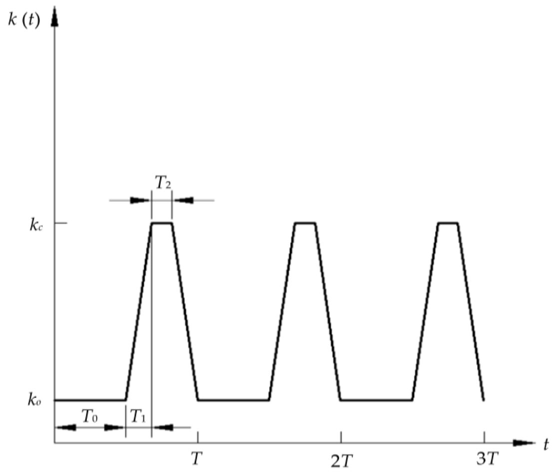


Figure 6: Variation in bending stiffness of the cracked beam [3]

Figure 6 depicts the impact of fracture depth on resonance frequencies and amplitudes using frequency response graphs. These results are the same as the ones from the nonlinear dynamic equation we talked about earlier. This shows how important it is to include nonlinearities in predictions about structures. The Lindstedt-Poincaré method turns complicated nonlinear differential equations into forms that can be solved by using an inadequate parameter α that stands for the oscillation amplitude [3]. In 2015, [15] applied the Lindstedt-Poincaré perturbation method to study the natural features of a hard-coating cantilever plate. In addition, such solutions were experimentally validated by FEM and measurement techniques. The authors have used it to understand the dynamics behavior of hard-coating damping structures and give judgment about the applicability of the perturbation method in the free vibration analysis of the hard-coating laminate plates. Following the natural frequencies and mode shapes calculation based on nonlinear elasticity plate theory, the Lindstedt–Poincaré perturbation method has been used. Approximate analytical results have been compared to the FEM obtained results and experimental measurements. The results are consistent with each other. This paper also reflects that the natural frequencies obtained by a linear analytical method have a huge gap with those obtained from the approximate analytical method. Thus, the study recommended that the Lindstedt-Poincaré perturbation method may be employed in the analysis of free vibration for hard-coating laminate plates.

As noted in [4], recent developments in the Lindstedt-Poincaré approach have greatly raised its accuracy and efficiency. For systems with higher-order nonlinearities, especially, a simplified method was presented that lowers computing cost while preserving great accuracy. This improvement makes the technique even more fit for contemporary engineering uses, where computational performance is absolutely vital. Additionally, [7] proposed a simplified Lindstedt-Poincaré approach that reduces pointless higher-order expansions and increases convergence rates without compromising accuracy. This approach performs rather well for considering systems with minor to intermediate nonlinearities since it fits well with numerical simulations. As part of microbeam analysis, [14] used the Lindstedt-Poincaré method to look into how cantilever microbeams react to superharmonic excitations. The work showed how things like thickness, length scale and activation forces can change the way the system reacts. The results revealed that despite lowering the response amplitude, raising the length scale parameter increases the superharmonic resonance frequency, therefore offering important information for microbeam design. Moreover [20] investigated the nonlinear vibrations of elastic beams including internal resonances using the Lindstedt-Poincaré approach. The work proved that the approach is a trustworthy instrument for evaluating systems with several degrees of freedom since it can effectively record intricate interactions between vibrational modes [20]. The Lindstedt-Poincaré method was used to simulate the nonlinear dynamics of energy-collecting cantilever beams with tip masses. The technique can maximize energy harvester designs since it accurately predicts frequency response even under high deflections [19]. Finally, [13] studied rotating machinery using the Lindstedt-Poincaré technique, where nonlinearities affect system stability. The technique predicted torsional vibrations and created damping solutions for high-speed rotating systems, confirming its applicability across engineering disciplines.

## 3. Galerkin Method

The Galerkin method has been one of the successful numerical methods for obtaining solutions of nonlinear differential equations for elastic beams, enabling one to deal with resultant nonlinear algebraic equations. It is frequently employed to estimate complex differential equations, such as those governing beam vibrations. It selects boundary-compliant trial functions to elucidate the matter. Recent research has shown that the Galerkin method not only simplifies the complex nonlinear differential equations controlling elastic beams but also improves the dependability of results in practical applications. Recent research published by [13, 21] solved the nonlinear differential equation of deflection of the elastic beam using the Galerkin technique [14], showing that the Galerkin approach allows nonlinear partial differential equations be transformed into ordinary differential equations, so greatly simplifying analysis, the use of this method to nonlinear beam deflection equations has been extensively tested. Furthermore, this work showed the efficient capturing of beam dynamics under different external stresses and boundary conditions by the Galerkin technique. The Galerkin technique was used to estimate a set of nonlinear algebraic equations, proving its efficiency in nonlinear situations. Their Galerkin technique for solving the differential problem yielded a trigonometric function with fitted coefficients in Chebyshev polynomials [13]. The third-order approximation matched the precise solution of the elliptic functions, proving the Galerkin method's efficacy and superiority for nonlinear differential equations. Confirming the method's stability, research by [4] showed how well it could solve nonlinearities in structural dynamics. They considered a fractional-order strongly nonlinear oscillator subject to random harmonic excitations:

$$\ddot{x} + \omega_o^2 x + \varepsilon^2 \mu D_t^\alpha x + \varepsilon g(x, \dot{x}) = \varepsilon^2 \xi(t) \quad (14)$$

where derivatives, ε, μ, and $\omega_0$ are nonnegative parameters, to be an odd function, and $D_t^\alpha$ $(x, \dot{x})$ represents the strongly nonlinear part, as well as they considered the following strongly nonlinear Duffing oscillator with fractional-order damping and random external harmonic excitation:

$$\ddot{x} + \omega_o^2 x + \varepsilon^2 \mu D_t^\alpha x + \varepsilon \delta x^3 = \varepsilon^2 \gamma cos(\Omega t + hW(t)) \quad (15)$$

where $\delta$ is the coefficient of the nonlinearity, $\gamma$ is the harmonic excitation intensity, $\Omega$ is the frequency the harmonic function and $W(t)$ is a standard Wiener process with the intensity $h$.

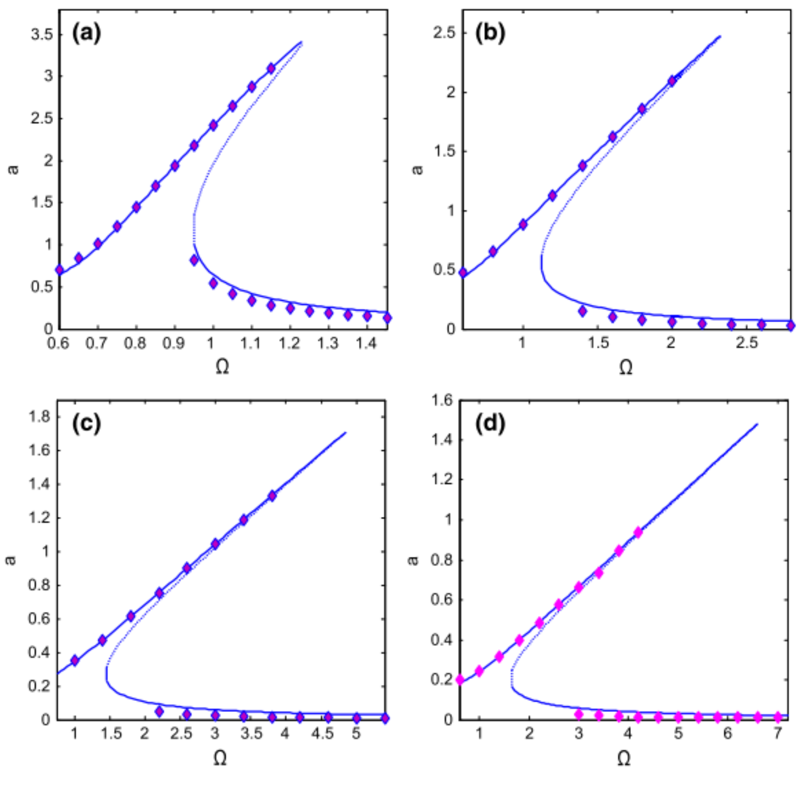


Figure 7: Frequency– amplitude response (a) δ = 1.0; (b) δ = 10; (c) δ = 100; (d) δ=250 [4]

Another researcher [22] employed this technology to project beam vibrations under harsh circumstances. This methodology enables scientists to obtain precise answers with minimal effort. It decomposes the complex task into more manageable, recognizable trial functions that satisfy the problem's boundary conditions. The objective at that stage is to reduce the "error" (the disparity between the actual response and the estimation). The second-order differential equation for nonlinear beam was conducted as;

$$N\left(x, y, y^k, y', y'', p\right) = 0 \quad (16)$$

Where; $x \in [a, b]$ and $y, y', y'', p, x, a\ and\ b$ are nonlinear terms, the approximate solution was assumed as:

(17) $$y = \sum_{n=0}^{M} A_n Y_n(x)$$

where $Y_n(x)$ satisfies the boundary conditions, and $A_n$ are coefficients to be determined with the limitations by an integer $M$. The research applied the Galerkin method:

(18) $$\int_a^b (x, y, y^k, y', y'', p) \sum_{n=0}^{M} \delta A_n Y_n(x)\, dx$$

Galerkin method-based boundary-compliant trial functions have been extensively used to approximate complex differential equations governing beam vibrations. By balancing the residual error over the pertinent area, weighting methods help to minimize the discrepancy between predicted and actual results. Research [22] verified in extreme conditions the effectiveness of beam vibrations in nonlinear systems. This method proved effective in the study of nonlinear forced vibration in aircraft wings with a rotating unbalanced mass, as outlined in [17]. The study addressed the nonlinear forced vibration of aircraft wings carrying a rotating unbalanced mass. To illustrate the application, Figure 8 depicts the deformation and undeformed wing coordinate system utilised in the work by [17], which represented the nonlinear forced vibration of aircraft wings. The graphic helps visualize the system and gives context for the boundary conditions and forces applied during the investigation.

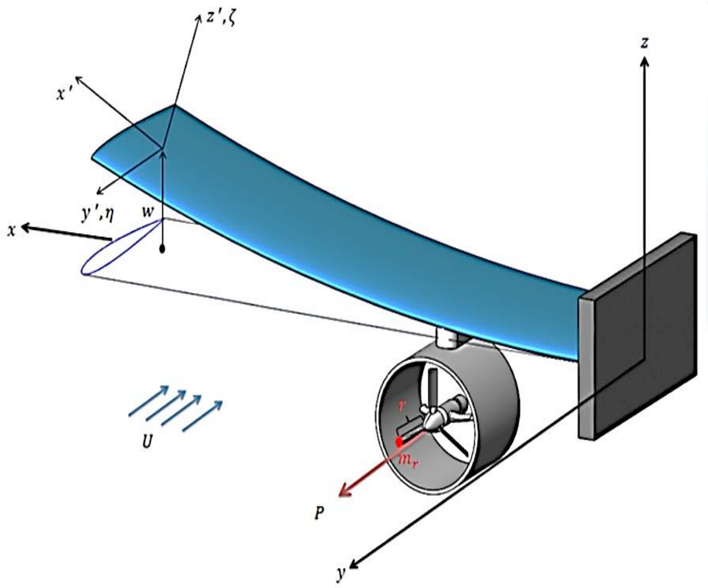


Figure 8: Coordinate Systems of the Wing: Deformed vs. Undeformed States [18]

The authors showed three-dimensional Cartesian $xyz$ and corresponding unit basis vectors $ijk$ are attached to the fixed reference framework $R$. Three Cartesian axes $x'y'z'$ and corresponding unit bases vectors i′j′k′ are attached to the wing framework $V$. Hamilton's variational principle was used to derive the equation of motion and applied boundary conditions using the Galerkin Method to derive the equation of force and deformation as:

(19) $$F_{em} = \left[m_r r\Omega^2 cos\Omega t \left(1 - \frac{w'^2}{2}\right) - m_r r\Omega^2 sin\Omega t w'\right] \hat{\imath} + [-m_r r\Omega^2 sin\Omega t\theta]\hat{\jmath} + \left[m_r r\Omega^2 cos\Omega t w' + m_r r\Omega^2 sin\Omega t \left(1 - \frac{w'^2}{2}\right)\right] \hat{k}$$

The study transferred the results of partial differential equations to ordinary nonlinear differential equations using the Galerkin method. They obtained the flutter occurrence possibility, frequency response function and multiple scales of the system in different primary resonance cases. Their study jumped between the two observed primary resonance cases. In addition to that non-resonant excitation, the stability of the system is independent of the rotating unbalanced mass parameter. This equation generally delineates the relationship between a beam's deflection and the elastic constant, mass, damping coefficient, and external forces. The issue is addressed by decomposing the principal partial differential equation into a more tractable set of ordinary differential equations using the Galerkin method. Applying the Galerkin technique, the deflection is represented as an accumulation of mode shapes and extended coordinates. This equation encapsulates the system's nonlinear dynamic nature, and in the Galerkin method for Elastic Beams in the method effectively approximates solutions to differential equations, as examined in research [12,13]. This is accomplished by rectifying or orthogonalizing the mistake relative to weighting functions throughout the problem domain. This strategy simplifies the intricate, unique differential problem into logarithmic or standard differential conditions. Researchers in [7] talked about how useful it is to combine Galerkin's method with easier perturbation methods like the Lindstedt-Poincaré method, which makes it even faster to use. A flexible tool in beam analysis, this hybrid technique described in [7] enables correct solutions even in the face of higher-order nonlinearities. A difficult topic is deconstructed into smaller, more comprehensible components. The Galerkin method has been widely used to solve nonlinear vibration issues in beam-like structures. When [13] calculated the elastic beam nonlinear reliable equation of deflection, this method rightly displayed large deflections and nonlinear dynamics. The method was more accurate and consistent, as their numerical outputs were the same. This work clarifies how the Galerkin method might solve challenging nonlinear situations, helping students and engineers. Even in [17], the Galerkin approach is explained as an efficient method for addressing the complexity of nonlinear differential equations. Moreover, [20] demonstrated that even complicated beam behaviors such as forced vibrations can be precisely described provided nonlinearities in the beam equations are adequately considered for utilizing Galerkin's technique. The fact that the obtained findings closely matched analytical solutions emphasises the dependability of the approach in forecasting beam responses [13, 16, 23]. Research by [3, 5, 24, 25] illustrates how this equation predicts beam vibrations, much like a recipe guides cooking. These researchers investigated several "ingredients" (such as cracks, strong forces, or moving supports) using the formula. They deduced the beam's behavior in practical settings, ensuring dependability, safety, and efficiency in construction. Another formulation of the equation is:

(20) $$m\ddot{w} + c\dot{w} + EI\frac{d^4w}{dx^4} - P\frac{d^2w}{dx^2} - + f(w) = F(t)$$

where the nonlinear effect of beam bending and material properties are captured by this equation. The external force is $F(t)$ and the simplified ordinary differential equations are solved numerically using the Runge-Kutta technique or finite element analysis to characterize and predict beam behavior under dynamic loads. The same attempt has been done by [26] their results showed in Figure 14. Accordingly, the study by [27] conducted the influences of the fluctuating rotating speed, the centrifugal force, the pre-twist angle and the pre-setting angle on the nonlinear dynamics of the rotating blade. This demonstrated its capacity to handle large deflections and nonlinearly moving structures. The results reveal that even in cases of significant nonlinearity, the technique may produce accurate and consistent forecasts. This extension of the governing equation forms a series of linked ordinary differential equations, making the problem numerically soluble. This equation is a generalized form of the Euler-Bernoulli beam equation with additional terms for damping, axial load and nonlinearity as shown in Figure 9. The governing equation for the rotating speed and presetting angle.

(21) $$\Omega(t) = \Omega_0 + \Omega_1 \cos(\omega t)$$

The torsion angle $\alpha$ changes linearly along the spanwise. The presetting angle of the plate root is $\alpha_0$, the tip is and $\alpha_1$and b is the plate length, so $\alpha = \alpha_0 + \frac{\delta}{b}x$, where $\delta = \alpha_1 - \alpha_0$

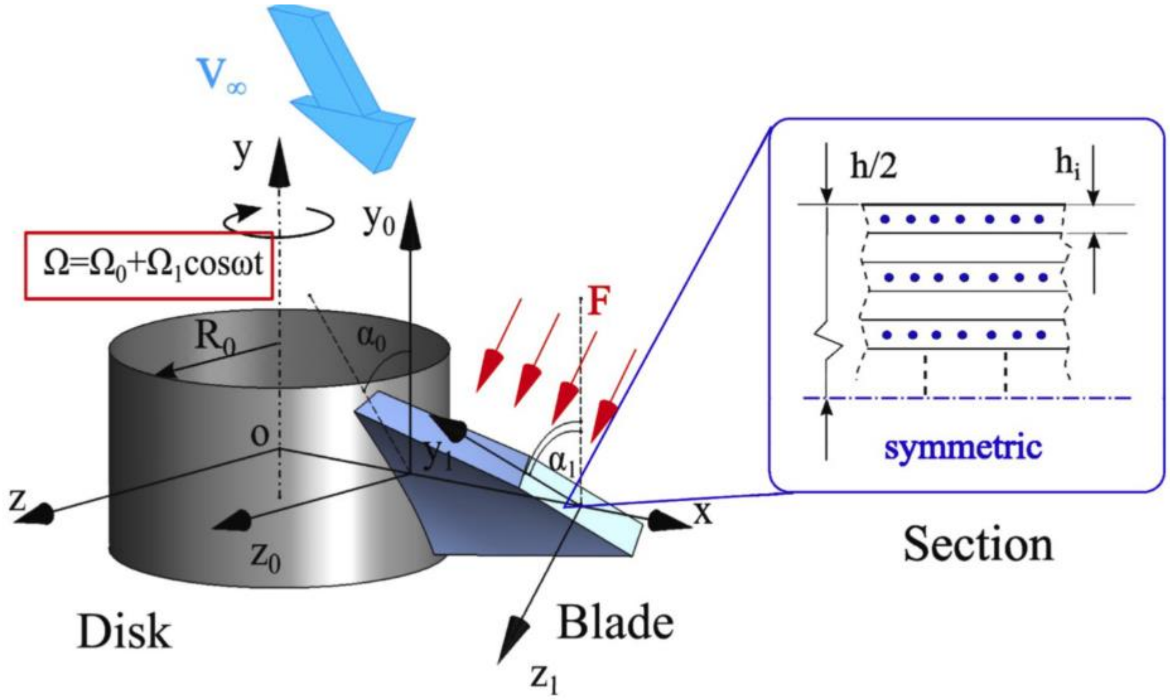


Figure 9: Blade model with variable speed [5]

The nonlinear ordinary differential governing equations of motion with two degrees of freedom are obtained by using the Galerkin method an example of their results is shown in Figure 15 which phase portrait associated with chaotic vibration behavior when the system is subjected to certain excitation parameters, notably at a frequency ratio of 0.5 (p = 0.5) and a specific rotational speed Ω. Structural dynamics and continuum mechanics allow for the development of the governing equation, balancing inertial, damping, stiffness and external forces. Implementing the Galerkin approach, complex equations may be simplified to ordinary differential equations, simplifying beam deflection analysis under diverse loading circumstances. Additionally, the study conducted a composite cantilever plate model with high-speed rotating blades that can be changed, analyzing many structural factors influencing the variation of blade natural frequency with speed change. The authors also adopted the Galerkin method with Chebyshev polynomials to obtain nonlinear ordinary differential equations of motion for a two-degree-of-freedom spinning blade. The authors developed a nonlinear wing structure model that incorporates a propeller system and coupled it with an incompressible unsteady aerodynamic model. They applied Hamilton's variational principle to derive the system governing equations and boundary conditions. Applying Galerkin's method, they reduced the obtained partial differential equation to conventional nonlinear differential equations for the mode shapes based on the rotational speed and frequency as shown in Figure 11 [27].

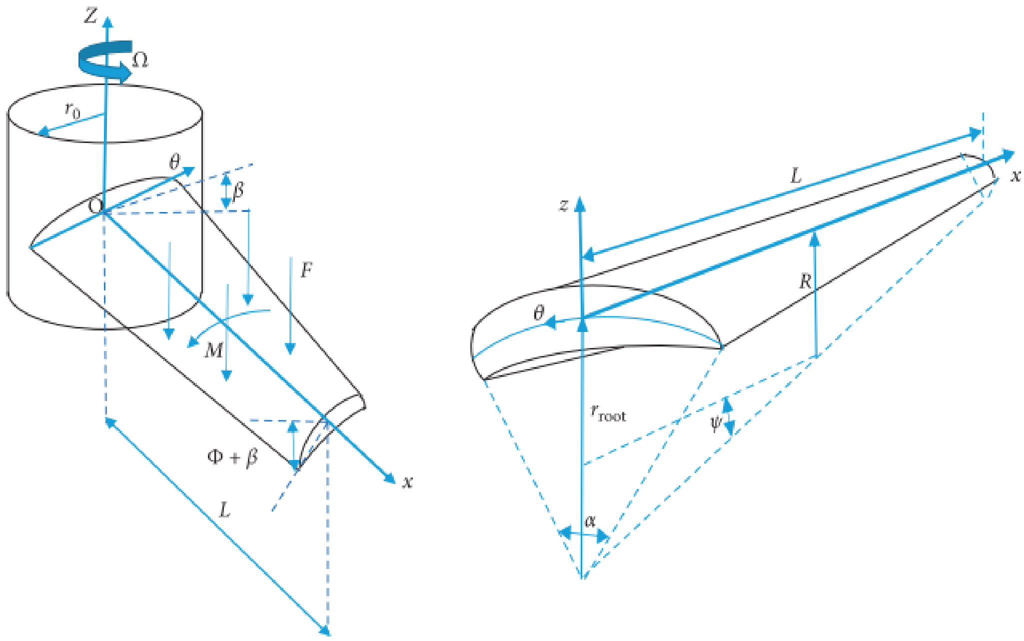


Figure 10: The model of presetting and pre-twisted cantilever conical shell with varying thickness [28]

The same attempt done by [28] used a Galerkin's method to convert the partial differential governing equations of motion to a set of nonlinear ordinary differential equations as shown in Figure 10. Analyzed the chaotic and periodic motions in the system, the phase portraits, time history diagrams, three-dimensional phase portraits, and power spectrum densities (PSD) were obtained. The approximate equations are derived as:

(22) $$u_o(x,\theta,t) = u_1(t)sin\left(\frac{\pi}{L}x\right)\cos(3\theta - 3x) + u_2(t)sin\left(\frac{3\pi}{L}x\right)\cos(\theta - x)$$

The same equation was derived for $v_o(x,\theta,t)$ and $w_o(x,\theta,t)$ then derived $\varphi_x(x,\theta,t)$ and $\varphi_\theta(x,\theta,t)$

(23) $$\varphi_x(x,\theta,t) = \varphi_{x1}(t)sin\left(\frac{\pi}{L}x\right)cos(3\theta - 3x) + \varphi_{x2}(t)sin\left(\frac{3\pi}{L}x\right)cos(\theta - x)$$

(24) $$F_1 = F_{11}sin\left(\frac{\pi}{L}x\right)cos(3\theta - 3x) + F_{22}(t)sin\left(\frac{3\pi}{L}x\right)cos(\theta - x)$$

(25) $$M_1 = M_{11}sin\left(\frac{\pi}{L}x\right)cos(3\theta - 3x) + M_{22}(t)sin\left(\frac{3\pi}{L}x\right)sin(\theta - x)$$

Using the equations above, the nonlinear equations of motion in terms of generalized displacements was obtained. All the inertia terms of $u_o$, $v_o$, and $\varphi_x$ in nonlinear equations of motion can be ignored since their influences are small compared to the inertia terms of $w_o$ and $\varphi_\theta$ . Then, derived the displacements $u_o$, $v_o$, and $\varphi_x$ with respect to the displacement $w_o$ and $\varphi_\theta$. In their analysis, they expressed $u_o$, $v_o$, and $\varphi_x$ in terms of $w_1$, $w_2$, $\varphi_{\theta 1}$, and $\varphi_{\theta 2}$ The detail is shown in [28].

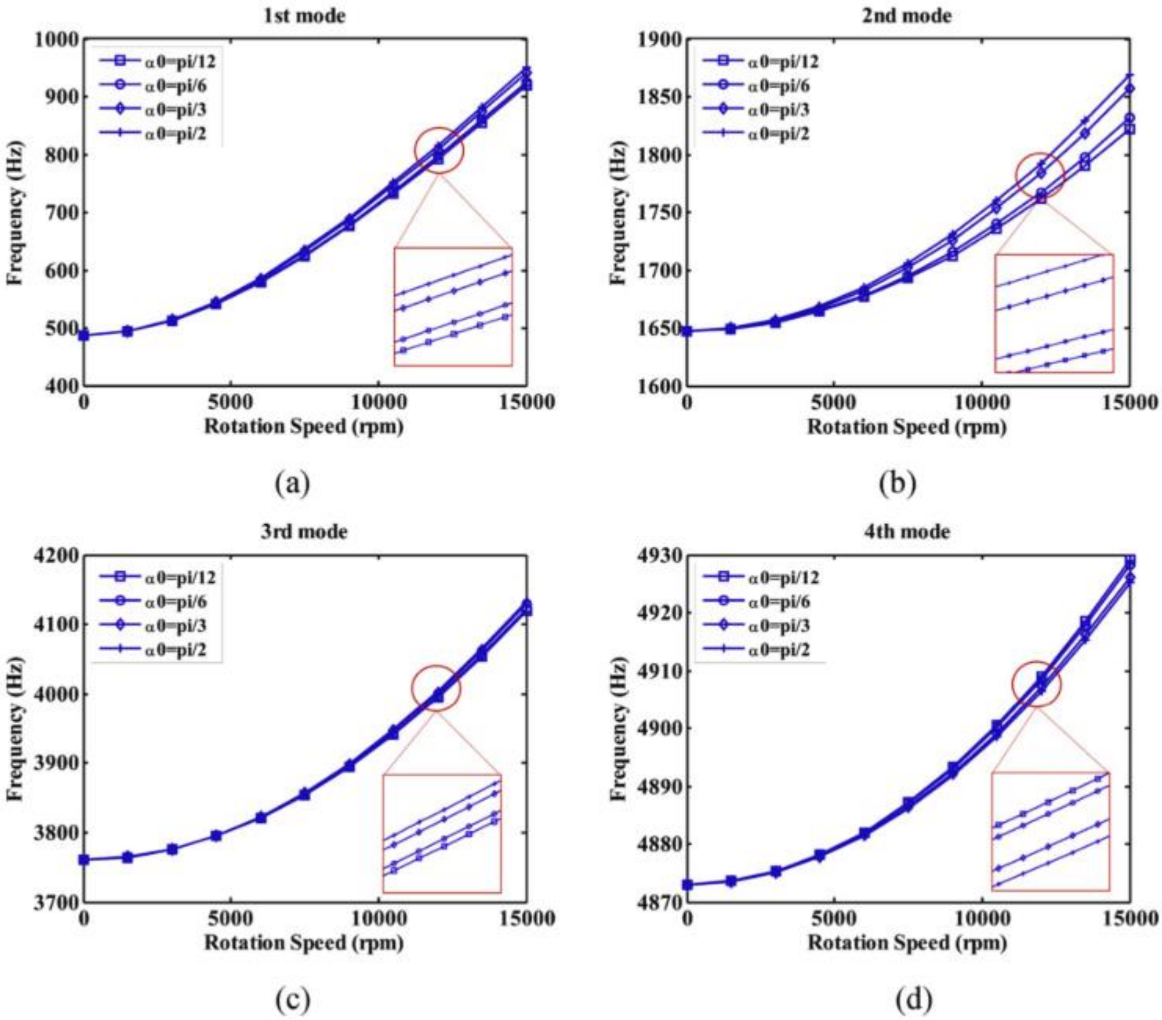


Figure 11: The 1st-4th mode with a different setting angle [27]

In 2007, a study by [29] used a Galerkin method to solve the uniform cantilever beam carrying a mass at the free end and exposed to sinusoidal base motion ( $y_g(t) = y_g sin(\Omega t)$). They assumed that the beam was initially straight, where L is the length, and $\rho A$ per unit length is a constant mass and constant stiffness. $E$ is Young's modulus of the material and $I$ is the principal cross-sectional area moments of inertia, and $\alpha$ is the orientation angle of the beam, s is used to denote arc-length along the beam, as shown in Figure 12.

(26) $$\rho A\ddot{w} + c\dot{w} + EI\{w'''' + (w'(w'w'')')'\} + \{[w'\rho A(L-s)]' + m(w')'\}\left(\ddot{y}_{gu} + g.\sin(\alpha)\right) - \frac{1}{2}\rho A\left[w'\int_s^l \frac{\partial^2}{\partial t^2}\int_0^s w'^2 ds ds\right]' - \frac{1}{2}m\left[w'\frac{\partial^2}{\partial t^2}\int_0^L w'^2 ds\right] + (\rho A + \delta[s-(L-\varepsilon)].m)\ddot{y}_{gv} = 0$$

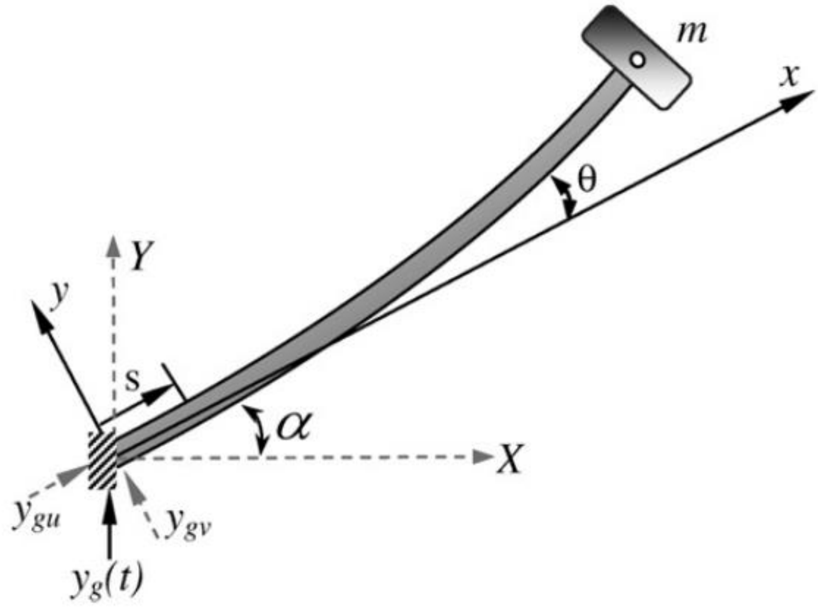


Figure 12: A schematic of the cantilever beam under consideration [29].

Substituting initial boundary conditions and simplifying the equation to use an approximate solution of the assumed equation, which was derived as:

(27) $$v(x,t) = \sum_n \emptyset_n(x) z_n(\tau)$$

Where; $\emptyset_n$ is the shape function of the n[th] linear mode, and $z_n$ is the time modulation of the n[th] mode. Then the governed equation for undamped linear free vibration under axial loading is governed by:

(28) $$\ddot{v} + v''' + \frac{\rho A g L^3}{EI} \sin'(\alpha)[(1-x)\,v'] + \frac{mgL^2}{EI}\sin(\alpha)\,.\,v'' = 0$$

Then, using a Galerkin method (equation 25) to acquire an ordinary differential equation form of a specified partial differential equation for an approximate solution. the truncated displacement function for the first mode becomes:

(29) $$v = \emptyset(x) z(\tau)$$

The beam ordinary differential equation was obtained as:

(30) $$h_1\ddot{z} + \mu h_1 \dot{z} + (h_2 + h_{10})z + h_5 z^3 - h_{11} z \frac{\partial^2}{\partial \tau^2}(z^2) - h_{12}\Omega_o^2 \sin(\Omega_o \tau)\cos(\alpha) - z.\,h_{13}\Omega_o^2 \sin(\Omega_o \tau)\sin(\alpha) = 0$$

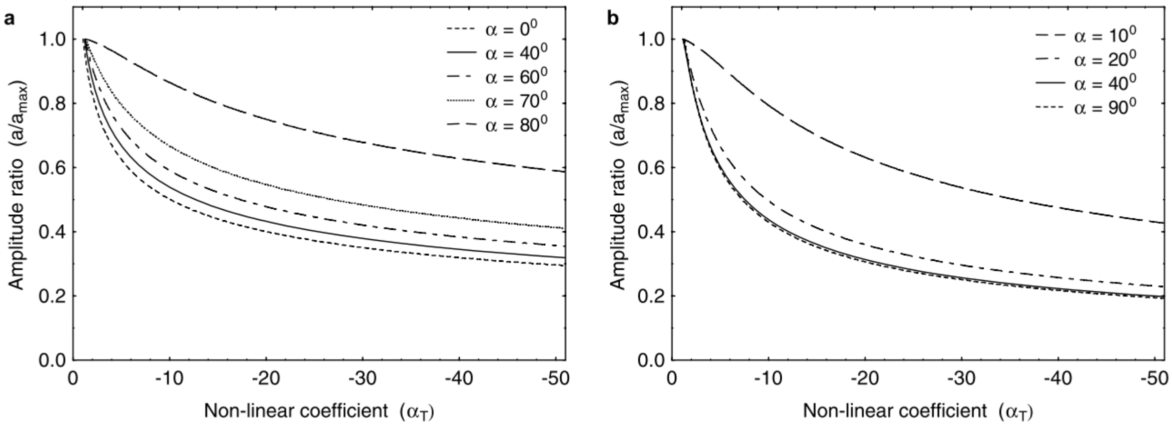


Figure 13: Variation of the steady-state amplitude ratio with the non-linearity coefficient for various orientation angles: $(a)\ y_g = 0.00025\ m, G = 1, c = 0.08, r = 0$; $(b)\ y_g = 0.03\ m, G = 1, c = 0.08, r = 0$ [30]

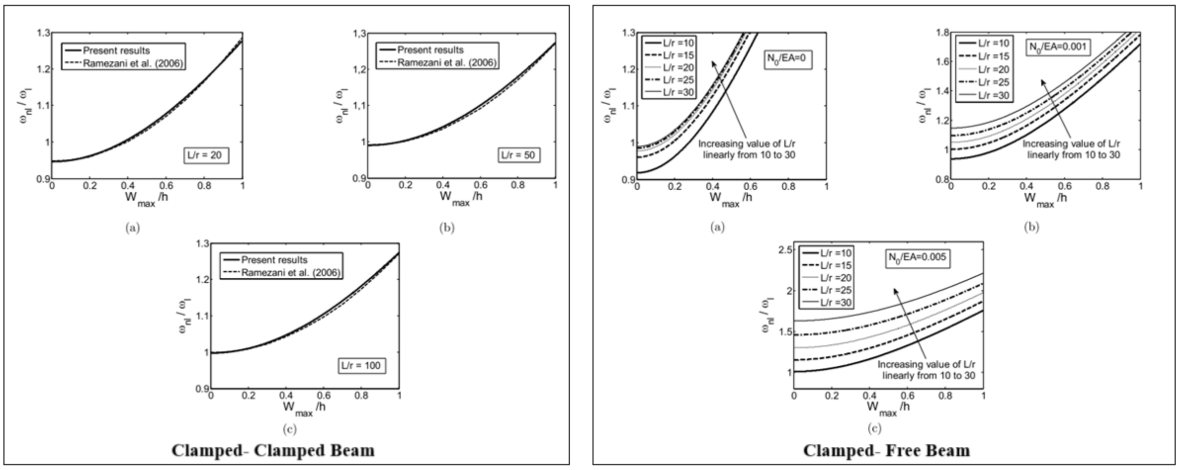


Figure 14. Nonlinear frequency of beam for varying slenderness ratios [27]

This approach is pragmatic and effective for analyzing the system's behavior in actual environments. The efficacy of the Galerkin technique extends to actual engineering applications, where it assists in recognizing probable structural problems and improving the efficiency of materials employed in design. The studies by [4,1] validate that it is a useful instrument for identifying and forecasting dynamic activities in practical surroundings. The results from [3, 5, 24, 25] show that these studies help engineers and scientists to detect damage by finding cracks or weaknesses before failure. Improve design for safer and more efficient materials that can handle vibrations. Predict behavior and understand how structures react under different circumstances. Substituting into the governing equation results in a system of interconnected ordinary differential equations that may be more readily assessed numerically. When substituted, the problem becomes a system of coupled ordinary differential equations that can be analyzed numerically with much less difficulty than the original system while still capturing the primary dynamics of the problem. Although it simplifies complicated nonlinear differential equations to ordinary differential equations (ODEs), to address these issues, researchers have devised meshless Galerkin systems [14]. For complex geometry, these substitutes improve computing performance and remove the need for advanced meshing. This approach uses generalised coordinates and mode expansions, hence mistakes in higher-order approximations could still happen. Accessing systems that are highly complex or possess intricate boundary conditions can be problematic, even if they function effectively. Studies, like those by [17], indicate that the Galerkin method may struggle with substantial nonlinearities or when the actual mode shapes significantly diverge from the expected trial functions. Researchers have developed meshless Galerkin algorithms to tackle these challenges [14]. These solutions eliminate the necessity for intricate meshing and enhance computational efficiency for complex geometry. This method employs generalised coordinates and mode expansions, which may cause errors in higher-order approximations, even so. Although the approach fairly effectively captures the primary dynamics, studies such as [13, 17] reveal that it cannot be able to manage strong nonlinearity or complex boundary conditions. As shown in [17], the choice of trial functions mostly defines the approximative efficacy of the Galerkin technique. When the real mode forms quite differently from the anticipated forms, this reliance on preset forms could cause convergence problems. Furthermore, the efficiency of the method in practical applications such as nonlinear vibrations of airplane wings [17] emphasises the need to couple with modern numerical techniques as Runge-Kutta, for best accuracy. Regardless of these challenges, concepts such as meshless Galerkin techniques [14] remove the necessity for meshing, therefore overcoming issues with complex geometry and hence increasing computational speed. Good forecasts and stability, on the other hand, depend on combining the Galerkin approach with additional computing methods in cases with substantial nonlinearity or dynamic fluctuations.

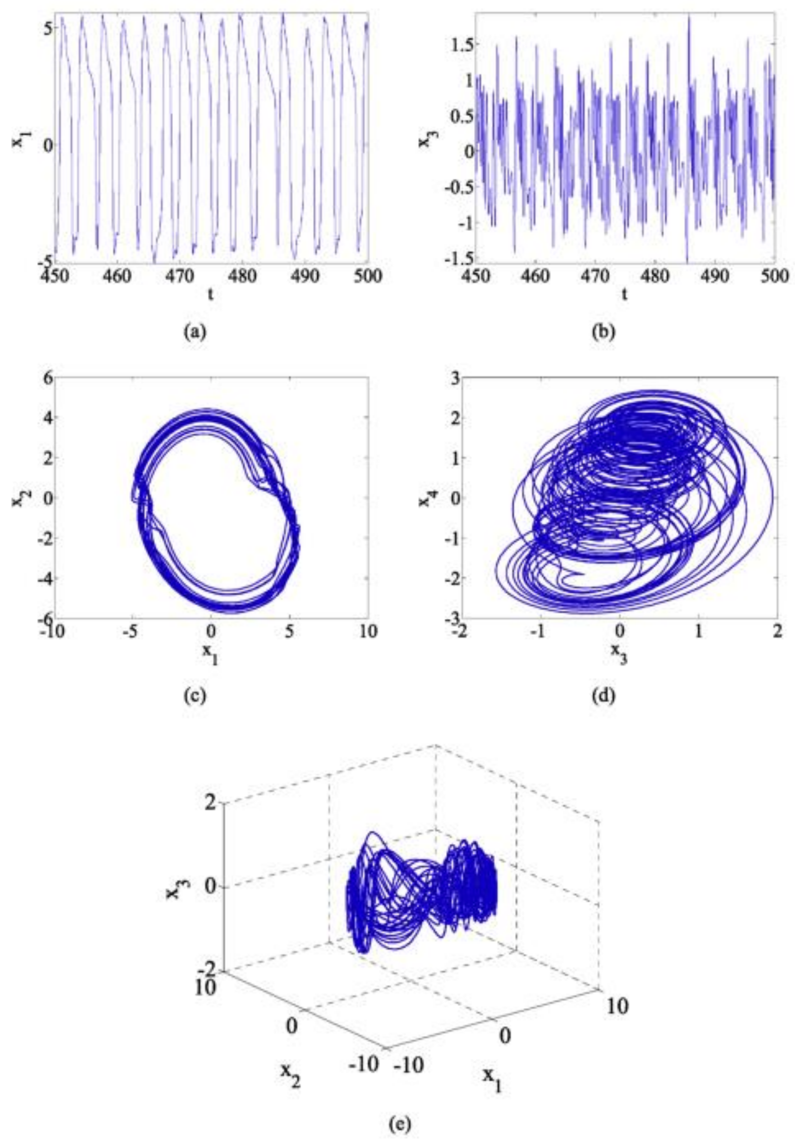


Figure 15: Chaotic Vibration Behavior When the System Is Subjected to Certain Excitation Parameters ($0{:}5 \leq p_{21} \leq 1$) [27]

As can be seen in Figure 13, the increase of the $\alpha$ in direct excitation reduces the performance of non-linearity coefficients on the decrease of the amplitude ratio. They concluded that the steady-state amplitudes diminish because of the non-linearity in both directly and parametrically excited systems. Nevertheless, the distinction of the orientation angle affects the performance of the non-linearity in the decrease of the amplitudes.

## 4. Discussion and Comparison

The complexities inherent in modelling nonlinear vibrations of beam-like structures, such as cantilever beams and plates, necessitate the application of sophisticated analytical and numerical methods capable of capturing the intricate behaviors induced by geometric and material nonlinearities. Among the prominent approaches are the Lindstedt-Poincaré perturbation method and the Galerkin technique, each exhibiting distinct advantages tailored to different regimes of nonlinearity.

### 4.1 Lindstedt-Poincaré Method:

This perturbation strategy leverages series expansions in a small parameter associated with the nonlinear amplitude or nonlinearity strength. It effectively transforms the nonlinear differential equations into hierarchically linearized forms, allowing rapid convergence and offering high computational efficiency. As evidenced in Figures 6 and 13, the method accurately predicts the variation of resonance frequencies and amplitude ratios as functions of nonlinearity coefficients and slenderness ratios. Particularly, Figure 13 illustrates how nonlinear frequencies evolve with changing slenderness ratios, with the method capturing shifts in natural frequencies and dynamic response characteristics. Similarly, Figure 12 demonstrates the dependence of amplitude ratios on the nonlinearity coefficient across various orientations. These results validate the method's proficiency in elucidating modest nonlinear effects but highlight its limitations in strongly nonlinear regimes where higher-order terms become significant.

### 4.2 Galerkin Method:

Conversely, the Galerkin approach provides a highly flexible, semi-analytical, and numerical framework suitable for dealing with large displacements, complex boundary conditions, and strong nonlinearities. By employing trial functions, often orthogonal polynomials or trigonometric functions, this technique reduces the governing equations to a set of coupled nonlinear algebraic equations. Figures 14 and 15, effectively visualize the detailed nonlinear dynamic behaviors, including the frequency response of composite blades at various rotational velocities and the variation of the steady-state amplitude ratio across different nonlinearity coefficients. Figure 14, in particular, depicts the nonlinear movement of a composite blade as the system's natural frequency shifts with increasing rotational speed, exemplifying the method's capability in simulating real-world complex vibrational phenomena. Moreover, Figures demonstrate the Galerkin method's accuracy in capturing mode coupling and bifurcation behavior under dynamic loads, crucial for structural integrity assessments.

### 4.3 Hybrid and Multi-Scale Techniques:

The integration of multiple-scales perturbation techniques with the Galerkin method, as discussed in, enhances the capacity to analyze resonance phenomena and complex modal interactions over extended time domains. Figure 2 depicts a comparison between frequency response curves obtained from analytical solutions, i.e., the Multiple-Scales Lindstedt-Poincaré (MSLP) approach, and numerical simulations using the fourth-order Runge-Kutta method. But as deflections increase, there are deviations in the Multiple-Scales (MS) method responses, which indicate the limitations of linear approximations. There is also added complexity due to variations in nonlinearity coefficients, as shown in Figure 3, where frequency response and large amplitude variations are observed. These variations mean that as nonlinear effects intensify, traditional methods can be inadequate, justifying the use of advanced analytical methods such as the MSLP approach to accurately capture system behavior. In addition, figures such as those in (not explicitly provided but conceptually similar to Figures 12 and 13) show the amplitude modulation and frequency shifts characteristic of nonlinear energy transfer, underscoring the advantages of hybrid methodologies in handling systems with moderate to strong nonlinearities.

### 4.4 Implications for Engineering Practice:

In high-stakes engineering applications—such as the vibrational analysis of aircraft wings subjected to fluctuating aerodynamic loads or the assessment of structural integrity in flexible robotic arms, the choice and combination of these mathematical tools are critical. Figures 6, 12, 13, and 14 collectively depict the progression from simplified approximate solutions to detailed numerical simulations, illustrating the importance of matching the method to the specific nonlinearity level and geometry of the problem. The figures emphasize that while the Lindstedt-Poincaré method excels in swift, first-order frequency prediction for systems exhibiting weak nonlinearities, the Galerkin method provides the detailed spatial and modal resolution necessary for designing resilient and failure-resistant structures. Continuous advancements in hybrid methodologies and computational capabilities promise to further bridge the gap between analytical simplicity and numerical precision, thus propelling the field toward high-fidelity predictive modelling essential for modern structural engineering.

Table 1: Comparative Performance of Approximate Methods in Literature

| **Aspect** | **Lindstedt-Poincaré Method** | **Galerkin Method** |
|---|---|---|
| **Strengths** | Effective for fixing issues with small nonlinear effects. | Very accurate for complex vibration problems in beams and plates. |
| | Methods with small vibrations or internal resonances benefit from understanding how little changes influence the system. | It effectively supports significant motions along with heavy loads, streamlining complex equations for more straightforward resolution. |
| **Weaknesses and limitations** | It's not very accurate for big changes or strong non-linear impacts. | It is essential to select functions judiciously to conform to the constraints of the situation. |
| | Faces difficulties with systems that have strong connections across modes | May need considerable time for highly intricate systems. |
| **ideal applications** | Systems with small vibrations or minor nonlinearities (e.g., small beam deflections) | Systems with large movements or heavy loads (e.g., airplane wings, rotating blades). |
| **Accuracy and Cost** | High (≤5% error). It costs less because it uses simple math methods. Multiple Scales (MS) is very accurate for small changes, but not as | Higher cost because it requires solving more complex equations, but it is highly accurate (≤2%) for complex systems, especially with the right setup. |

| | | |
|---|---|---|
| | accurate for big moves as MSLP (Hybrid) (≤1%). | |
| **Computational Efficiency** | Very High (Low iteration | Medium |
| **Adaptability in Managing The absence of Linearity** | This method works great for finding small nonlinearities in the way a system behaves, but it has trouble with bigger or more important changes, like significant deflections or vibration shifts. | Even though it requires more effort and time, the Galerkin approach is extremely effective when working with complex, nonlinear systems. This demonstrates how adaptable it is at recording complex behavior in many contexts. |

*Sources:* [6], [9], [11], [13], [17], [22], [28]

## 5. Conclusions

Non-linear beam vibration under time-varying boundary conditions may happen when the boundaries of a beam are constrained to undergo displacements in time variation. A beam may experience this in complex structures where a structural element interfaces with another, or when the beam itself is subjected to time-varying external pressure. The nonlinear vibration arises from the immense amplitudes in the beam that may result in extreme variations of the deflection and response of the beam. The summary of the review is presented below:

- The Lindstedt-Poincaré method offers efficient and accurate approximations for small to moderate nonlinearities, making it highly suitable for initial assessments and dynamic response estimations.
- The Galerkin approach demonstrates greater accuracy in modeling large displacements, dynamic stresses, and intricate boundary conditions, enhancing reliability in detailed structural analyses.
- The hybridization of these methods reduces computational complexity and processing time, enabling practical solutions for complex nonlinear systems without sacrificing accuracy.
- Both techniques are adaptable to various real-world scenarios, including aircraft wings, flexible robots, and beams with time-varying conditions, illustrating their robustness and versatility.
- Integrating Lindstedt-Poincaré and Galerkin methods facilitates efficient, high-fidelity modelling of nonlinear vibrations, ensuring precise predictions essential for the safe and optimal design of modern structures.

### 5.1 Future Directions and Applications:

Combining the Lindstedt-Poincaré and Galerkin methods provides a useful method to address difficult nonlinear vibration problems in various types of real-world constructions. The Lindstedt-Poincaré method provides efficient and approximate solutions for small to moderate nonlinearities, especially. Conversely, the Galerkin approach performs best with more complex systems with big displacements and dynamic loads, therefore providing relatively precise results. The simulation of real engineering structures like aerodynamic wings, elastic robots, and time-varying loaded and constrained structures can be greatly improved by this supplementing power of the methods.

### 5.2 Limitations of the Existing Methods:

Although both methods consider nonlinear vibrations, they might not be suitable for quite random or rather nonlinear signals. Researchers should concentrate on improving present methods or creating blended solutions that are more suitable for these kinds of problems going forward.

### 5.3 Potential for Hybrid Methods:

Combining the Galerkin approach with perturbation approaches such as the Lindstedt-Poincaré method could provide a possible avenue for more exact and computationally efficient solutions for nonlinear vibrations, especially in highly dynamic or complicated systems. This approach may allow for more effective capturing of subtle events in complex structural systems.

## Conflict of Interest

The author declares that there is no conflict of interest.

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