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Shifted Chebyshev-Based Methods for Solution of Nonlinear Differential Equations

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Abstract: In this study, we have used the Shifted Chebyshev technique to solve nonlinear ODEs. The method is to rewrite the problem in a more stable form using Shifted Chebyshev polynomials, which have proven efficient and accurate in numerical solutions. The application of this method enables dealing with sophisticated nonlinear equations and makes it easy to find approximate solutions within a short time. The paper points out the advantages of the method in question, namely, diminished computational needs and better accuracy, and is exemplified by practical exercises. The current analysis concerning solving nonlinear ordinary differential equations using the Shifted Chebyshev technique offers a straightforward and readily applicable solution. Numerical results obtained from the proposed method show the precise agreement with the exact solution with minimum errors that outperforms the existing methods. Several examples are solved to show the effectiveness of the proposed method.

Keywords: *Chebyshev Polynomials; Shifted Chebyshev Polynomial; Ordinary Differential Equation, Nonlinear Equations, Numerical Aprroximation.*

1. Introduction

Differential equations are essential to mathematics, they serve a very important function, that is, deriving and describing algorithms for system dynamics irrespective of the field concerned. Each equation tells a story of relationships between functions and their respective change rates, making it easy to track transformation and forecast outlook [1]. Be it the swinging of a simple pendulum or the complex behavior of a weather system, differential equations serve an essential function in disciplines such as physics biology engineering and economics.

For the past few centuries, the study of differential equations has gone through significant changes, and it has resulted in a robust theoretical base and exhaustive numerical methodologies [2]. While the classical methods emphasize the obtaining of accurate results for specific equations, the more recent developments focus on problems that cannot be solved through analytic means. This balance of underpinning theory and practice illustrates the unsophisticated techniques and uses of differential equations in mathematics and abstract phenomena in the physical universe.

However, the utilization of differential equations is still limited due to insufficient methods for solving and interpreting them accurately, particularly with the presence of non-linearity, extensive touching parameters, or disorderly behaviors [3]. This problem presents a great opportunity because the development of efficient and accurate methods to solve these equations will always be at the forefront of investigation. Nonlinear differential equations provide particular difficulties because of their intricacy and sensitivity to beginning conditions [4]. Bifurcations, chaos, and multi-stability are common

phenomena of nonlinear systems, in contrast to linear equations, where superposition concepts are applicable. These features call for creative numerical techniques to guarantee accuracy, stability, and computational effectiveness.
Among the different types of differential equations, ordinary differential equations (ODEs) are a basic subclass of differential equations that deal with functions of a single independent variable and their derivatives. ODEs are used to simulate processes in which a quantity's rate of change is dependent on that quantity, as in population growth models, electrical circuits, and mechanical systems [5]. One of the main areas of study in both pure and applied mathematics is ODEs.
Many real-world issues result in complicated, nonlinear, or high-dimensional systems that do not accept closed-form solutions, even if some ODEs can be solved analytically. This has prompted the creation of numerical techniques to provide highly accurate approximations of solutions, enabling the analysis of previously unsolvable systems [6].
Chebyshev polynomials are a class of orthogonal polynomials that have found extensive applications in numerical analysis, particularly in the solution of ordinary differential equations (ODEs). Defined on the interval $[-1,1]$, these polynomials exhibit properties such as orthogonality, minimal oscillation, and efficient computation, making them an ideal choice for constructing numerical methods [7]. Among their many applications, Chebyshev polynomials are widely used in spectral methods, collocation methods, and pseudospectral methods for approximating solutions to ODEs with high accuracy [8].
One of the key advantages of Chebyshev polynomials is their connection to the roots and extrema of $T_n(x)$, which provide optimal points for interpolation and numerical stability [9]. These roots are often used as collocation points in numerical schemes, ensuring efficient error distribution and avoiding phenomena such as over fitting or instability. Moreover, the polynomials' recursive structure and ability to approximate smooth functions with exponential convergence underlie their effectiveness in computational settings [10].
The role of Chebyshev polynomials in solving ODEs is not merely computational but also analytical [11]. They arise naturally as solutions to the Sturm-Liouville problem and possess a solid theoretical foundation that guarantees orthogonality and eigenfunction properties [12]. This dual nature of Chebyshev polynomials—both as mathematical objects and practical tools—makes them invaluable in developing robust numerical techniques [13].
In modern numerical methods, Chebyshev polynomials are employed to transform ODEs into systems of algebraic equations, which can then be solved using standard numerical techniques. Their use ensures that the approximations are not only accurate but also computationally efficient, particularly for problems requiring high-order solutions or spectral accuracy. Thus, Chebyshev polynomials stand at the intersection of mathematical elegance and practical utility, enabling precise and efficient solutions to a wide range of ODE problems. In 1854, the author in [10] introduced the Chebyshev polynomials for the first time. Chebyshev, in fact, was a mathematician who initially made the analysis of this polynomial widely known. Chebishev polynomials can be used in mathematical analysis to characterize analytical functions and solve the Fredholm integral problem [14], [15]. In cryptography, these polynomials are also quite effective at solving encryption keys proposed [16] using Chebyshev polynomials in statistics techniques like Maximum Likelihood Estimation (MLE) to estimate a parameter [17]. There are more uses for the Chebyshev polynomials in various studies, including [18],[19],[20],[21],[22].
The authors in [23],[24],[25], presented a new analytical approach that can be use to solve nonlinear partial differential equation (NPDE) and Nonlinear schrodinger equation (NLSE) [26],[27], the proposed method can also be use to solve equation the NPDEs and NLSEs. From the practical point of view and application perspective, the proposed method has unlimited application in control theory [28],[29],[30]. The authors present a novel technique for commutativity of linear time-varying

systems in [31],[32],[33]. This method has the potential to simplify design and analysis. The author in [34] extend the classical Chebyshev polynomials of the first and second kinds by employing the framework and operational methods associated with Kampé de Fériet-type Hermite polynomials. The approach allows the establish integral representations for the generalized Chebyshev polynomials. Moreover, the authors in [35] employ the integral representation approach to demonstrate the connection between classical Hermite polynomials and a newly introduced family of Hermite polynomials related to parabolic cylinder functions. To tackle the nonlinear ODEs and PDEs problem, the authors [36],[37],[38] investigated and come up with different techniques that lead to solution.

2. Preliminaries

This section introduces definitions, basic properties along side with some formulas of Chebyshev polynomials that will be use to support the proposed method.

2.1. Chebyshev polynomials

Definition 2.1.1. *If* $t = \cos\theta (0 \leqslant \theta \leqslant \pi)$*, the function*

(1) $$T_n(t) = \cos(n\theta) = \cos(n\, arc\cos t),$$

is a polynomial of t *of degree* $n (n = 0, 1, r, \ldots)$*.* T_n *is called the chebyshev polynomial of degree n or the Chebyshev polynomials of the first kind. when* θ *increases from 0 to* π*,* t *decreases from 1 to -1 . Then the interval* $[-1, 1]$ *is domain of* $T_n(t)$*.*

Definition 2.1.2. *And the function*

(2) $$U_n(t) = \sin\left(n\cos^{-1} t\right),$$

is the Chebyshev polynomials of the second kind.

The Chebshev polynomials of the first and second kind are denoted by $T_n(x)$ and $U_n(x)$ respectively. The subscript n is the degree of these polynomials. The Chebshev polynomials of the first and second kind are closely related. For example, a Chebyshev polynomial of first kind can be represented as a linear combination of two Chebyshev polynomials of second kind,

$$T_n(x) = \frac{1}{2}\left(U_n(x) - U_{n-2}(x)\right),$$

and the derivative of a Chebyshev polynomial of first kind can be written in terms of a Chebyshev polynomial of second kind,

$$T_n'(x) = nU_{n-1}(x), n = 1, 2, \cdots$$

By considering the Chebyshev polynomials of first kind and we use them for approximating a function and a particular solution for ODEs. The solution of the differential equation

(3) $$\left(1 - x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2 y = 0,$$

is called the Chebyshev polynomial. The Chebyshev polynomial of first kind is,

$$T_n(x) = y = \cos\left(n\cos^{-1} x\right).$$

To verify, $y=\cos\left(n\cos^{-1}x\right)$ satisfies the differential equation (3)

$$\begin{aligned}y&=\cos\left(n\cos^{-1}x\right),\\ \frac{dy}{dx}&=\frac{n}{\sqrt{(1-x^2)}}\sin\left(n\cos^{-1}x\right),\\ \sqrt{(1-x^2)}\frac{dy}{dx}&=n\sin\left(n\cos^{-1}x\right).\end{aligned}$$

Differentiating both sides with respect to x, we have

$$\sqrt{(1-x^2)}\frac{d^2y}{dx^2}-\frac{x}{\sqrt{(1-x^2)}}\frac{dy}{dx}=-n^2\frac{\cos(n\cos x)}{\sqrt{(1-x^2)}}.$$

Multiplying both sides by $\sqrt{(1-x^2)}$, we get

$$\begin{aligned}\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}&=-n^2\cos\left(n\cos^{-1}x\right),\\ \left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{d}{dx}+n^2y&=0.\end{aligned}$$

Therefore, $y=T_n(x)=\cos\left(n\cos^{-1}x\right)$ is the solution of the differential equation (3).

2.2. Chebyshev polynomials functions

The first fifth polynomials of the first kind are denoted by T_n and are listed below:

$$\begin{aligned}n=0,T_0(x)&=\cos 0=1\\ T_1(x)&=\cos\left(\cos^{-1}x\right)=x\\ T_2(x)&=\cos(2\theta),\ \text{where }\theta=\cos^{-1}x,x=\cos\theta\\ &=2\cos^2\theta-1=2x^2-1\\ T_3(x)&=\cos 3\theta=4\cos^3\theta-3\cos\theta=4x^3-3x\\ T_4(x)&=\cos 4\theta=2\cos^2 2\theta-1=2\left(2x^2-1\right)^2-1\\ &=8x^4-8x^2+1.\end{aligned}$$

Also the first fifth of second are:

$$\begin{aligned}&U_0(x)=\sin 0=0\\ &U_1(x)=\sin\left(\cos^{-1}x\right)=\sin\theta=\sqrt{(1-x^2)}\\ &U_2(x)=\sin(2\theta),\text{ where }\theta=\cos^{-1}x,x=\cos\theta\\ &=2\sin\theta\cos\theta=2x\sqrt{(1-x^2)}\\ &U_3(x)=\sin 3\theta=3\sin\theta-4\sin^3\theta=\sin\theta\left(3-4\sin^2\theta\right)\\ &=\sqrt{(1-x^2)}\left(4x^2-1\right)\\ &U_4(x)=\sin 4\theta=2\cos 2\theta\sin 2\theta=\sqrt{(1-x^2)}\left(8x^3-4x\right).\end{aligned}$$

2.3. Matrix representation

The Chebyshev first kind polynomial can be represented in matrix form as $T_n(x)$ that satisfies the determinant.

$$\begin{vmatrix} x & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2x & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2x & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2x & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2x & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2x & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2x \end{vmatrix}.$$

For $n = 2$

$$T_2(x) = \begin{vmatrix} x & 1 \\ 1 & 2x \end{vmatrix} = 2x^2 - 1.$$

For $n = 3T_3(x) = \begin{vmatrix} x & 1 & 0 \\ 1 & 2x & 1 \\ 0 & 1 & 2x \end{vmatrix} = 4x^3 - 3x.$

For $n = 4T_4(x) = \begin{vmatrix} x & 1 & 0 & 0 \\ 1 & 2x & 1 & 0 \\ 0 & 1 & 2x & 1 \\ 0 & 0 & 1 & 2x \end{vmatrix}.$

$$= x\left(8x^3 - 4x\right) - \left(4x^2 - 1\right) = 8x^4 - 8x^2 + 1.$$

231. Properties of the Chebyshev Polynomials

Property 1. Generating Function The function which generates the Chebyshev polynomials is

$$\frac{(1-2x)}{(1-2xz+z^2)} \sum_{0}^{\infty} z^2 T_s(x) = \frac{(1-z)}{(1-2xz+z^2)},$$

i.e the coefficient of z^n in the expansion of $\frac{(1-a)}{(1-22z+z^2)}$ is $T_s(x)$.

Property 2. Orthogonality of $T_g(x)$

(i) $\int_{-1}^{1} \frac{T_n(x)T_n(x)dx}{\sqrt{1-x^2}} = 0$ if $m = n$
(ii) $\int_{-1}^{1} \frac{T_n(x)T_n(x)dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$ if $m = n \neq 0$
(iii) $\int_{-1}^{1} \frac{T_n(x)T_n(x)dx}{\sqrt{1-x^2}} = \pi, m = n = 0.$

Property 3. Recurrence Relations

1. $T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x)$

Proof: Put

$$\begin{aligned}
T_n(x) &= \cos\left(n\cos^{-1}x\right)\\
\cos^{-1}x &= \theta, \cos\theta = x\\
T_n(x) &= \cos(n\theta)\\
T_{n+1}(x) &= \cos(n+1)\theta\\
&= \cos n\theta\cos\theta - \sin n\theta\sin\theta\\
T_{n-1}(x) &= \cos(n-1)\theta\\
&= \cos n\theta\cos\theta + \sin n\theta\sin\theta.
\end{aligned}$$

Adding $\quad T_{n+1}(x) + T_{n-1}(x) = 2\cos n\theta\cos\theta$

$$= 2xT_n(x) \quad [\cos\theta = x]$$

$\therefore \quad T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x).$

2. $\left(1-x^2\right)T_n{}'(x) = nT_{n-1}(x) - nxT_n(x).$

Polynomials in terms of $T_n(x)$

$$\begin{aligned}
T_0(x) &= 1\\
T_1(x) &= x\\
T_2(x) &= 2x^2 - 1\\
x^2 &= \frac{T_2(x) + T_0(x)}{2}\\
T_3(x) &= 4x^3 - 3x\\
x^3 &= \frac{T_3(x) + 3T_1(x)}{4}\\
T_4(x) &= 8x^4 - 8x^2 + 1\\
x^4 &= \frac{T_4(x) + 4T_2(x) + 3T_0(x)}{8}.
\end{aligned}$$

Second Kind.
3.

$$U_{n+1}(x) + U_{n-1}(x) = 2xU_n(x).$$

Proof:

$$U_n(x) = \sin\left(n\cos^{-1}x\right).$$

Put

$$
\begin{aligned}
\cos^{-1} x &= \theta, \cos\theta = x \\
U_n(x) &= \sin(n\theta) \\
U_{n+1}(x) &= \sin(n+1)\theta \\
&= \sin n\theta \cos\theta + \cos n\theta \sin\theta \\
U_{n-1}(x) &= \sin(n-1)\theta \\
&= \sin n\theta \cos\theta - \cos n\theta \sin\theta \\
+T_{n-1}(x) &= 2\sin n\theta \cos\theta \\
&= 2xU_n(x) \quad [\cos\theta = x]
\end{aligned}
$$

$$
\begin{aligned}
\text{Adding} \qquad T_{n+1}(x) + T_{n-1}(x) &= 2\sin n\theta\cos\theta \\
&= 2xU_n(x) \\
\therefore \qquad U_{n+1}(x) + U_{n-1}(x) &= 2xU_n(x)
\end{aligned}
$$

4.

$$\left(1-x^2\right) U_n'(x) = nxU_n(x) - nU_{n-1}(x).$$

Remark: Chebyshev polynomials, particularly those of the first kind, are widely utilized in approximation theory due to their minimax property, which ensures the smallest maximum deviation from zero among all polynomials of a given degree with leading coefficient one. This makes them especially valuable in numerical analysis and spectral methods, as they provide optimal nodes for interpolation and reduce Runge's phenomenon. Their recursive structure and orthogonality on the interval with respect to the weight function further contribute to their efficiency and analytical convenience in solving differential equations and constructing polynomial approximations.

3. Shifted Chebyshev Polynomial

3.1. Overview of Shifted Chebyshev Polynomial

Nonlinear differential equations are the basic tools in describing various physical phenomena, from chemical reactions and spring-mass systems to the bending of beams. They are also found in ecological and economic studies. The most prominent representatives from the family include Riccati, Emden-Fowler, Duffing, Van der Pol, Rayleigh, and Yermakov equations. The greater part of nonlinear differential equations cannot be solved exactly, and one has to resort to different methods of approximate and numerical solving. Previous other numerical approaches for the nonlinear differential system include the Adomian decomposition method [39], [40] a linearization method of [41] sixth-degree B-spline method [42], decomposition method [13], and presently by the HPM [44]. The researchers in [45] have also taken into account the use of the quasilinearization methods to solve the nonlinear problems in physics. Reference [46] takes the Chebyshev collocation matrix process in arriving at solutions for nonlinear differential equations. The problem reduces to a matrix equation representing a most powerful system of nonlinear algebraic equations with unknown Chebyshev coefficients of a special form associated with the Chebyshev collocation points. The least square methods was introduced in [47], [48]. The authors in [49] investigate the shifted Chebyshev method for solving high order ODEs.
This paper discusses some changes and corrections to the use of the shifted Chebyshev collocation method to solve general forms of nonlinear differential equations.

$$\sum_{k=0}^{m}\sum_{s=0}^{n} Q_{k,s}(x)y^{s}(x)y^{(k)}(x) + \sum_{k=1}^{m}\sum_{s=1}^{m} P_{k,s}(x)y^{(s)}(x)y^{(k)}(x) = f(x), \tag{4}$$

with conditions

$$\sum_{k=0}^{m-1}\left(a_{ik}y^{(k)}(a)+b_{ik}y^{(k)}(b)+c_{ik}y^{(k)}(c)\right)=\alpha_i, \quad i=0,1,\ldots,m-1, \tag{5}$$

where $y^{(0)}(x)=y(x)$ and $y(x)\in C^m[0,L]$ is an unknown function. $Q_{k,s}(x), P_{k,s}(x)$ and $f(x)$ known functions which are defined on $[0,L]$.
We shall view the numerical solution of Eq. (4) from Eq. (5) in the form of truncated shifted Chebyshev series

$$y_N(x)=\sum_{r=0}^{N}a_rT^*_{L,r}(x), \quad x\in[0,L], \tag{6}$$

where N can take any positive integer so that $N>m$. We use notation $T^*_{L,r}(x)$, whereas $r=0,1,\ldots,N$ for shifted Chebyshev polynomials, which can be derived also from the following recurrence relationship $[9,10]$:

$$T^*_{L,r+1}(x)=2\left(\frac{2x}{L}-1\right)T^*_{L,r}(x)-T^*_{L,r-1}(x), r=1,2,\ldots,$$

where $T^*_{L,0}(x)=1, T^*_{L,1}(x)=\frac{2x}{L}-1$. The exact form of the shifted Chebyshev polynomilas $T^*_{L,r}(x)$ of degree r is given as

$$T^*_{L,r}(x)=r\sum_{p=0}^{r}(-1)^{r-p}\frac{(r+p-1)!2^{2p}}{(r-p)!(2p)!Lp}x^p, \tag{7}$$

where $T^*_{L,r}(0)=(-1)^r$ and $T^*_{L,r}(L)=1$. The orthogonality condition is

$$\int_0^L T^*_{L,j}(x)T^*_{L,i}(x)w_L(x)dx=h_k\delta_{ji,}$$

where $w_L(x)=\left(Lx-x^2\right)^{-1/2}$ and $h_i=b_i\pi/2, b_0=2, b_i=1, k\geq 1$. By Eq. (4), we have the k-th derivatives of $T^*_{L,r}(x)$

$$\left(T^*_{L,r}\right)^{(k)}(x)=T^{*,k}_{L,r}(x)=r\sum_{p=m}^{r}(-1)^{r-p}p(p-1)\ldots(p-k+1)\frac{(r+p-1)!2^{2p}}{(r-p)!(2p)!L^k}x^{p-k} \tag{8}$$

where $p\geq m-1$.

3.2. Solution Method

Using Eq. (36), we have the ks-th derivatives of approximate solution $y_N(x)$

$$y_N^{(k)}(x)=\sum_{r=0}^{N}a_r\left(T^*_{L,r}\right)^{(k)}(x)=\sum_{r=k}^{N}a_rT^{*,k}_{L,r}(x). \tag{9}$$

From (4), (6) and (9), we have

(10)
$$\begin{aligned}&\sum_{k=0}^{m}\sum_{s=0}^{n}Q_{k,s}(x)\left(\sum_{r=0}^{N}a_rT_{L,r}^{*}(x)\right)^{s}\left(\sum_{r=k}^{N}a_rT_{L,r}^{*,k}\right)\\&+\sum_{k=1}^{m}\sum_{s=1}^{m}P_{k,s}(x)\left(\sum_{r=s}^{N}a_rT_{L,r}^{*,s}\right)\left(\sum_{r=k}^{N}a_rT_{L,r}^{*,k}\right)=f(x).\end{aligned}$$

collocating Eq. (10) at $N-m+1$ points $x_p, p=0,1,\ldots,N-m$ leads to

(11)
$$\sum_{k=0}^{m}\sum_{s=0}^{n}Q_{k,s}\left(x_q\right)\left(\sum_{r=0}^{N}a_rT_{L,r}^{*}\left(x_q\right)\right)^{s}\left(\sum_{r=k}^{N}a_rT_{L,r}^{*,k}\left(x_q\right)\right)$$

(12)
$$+\sum_{k=1}^{m}\sum_{s=1}^{m}P_{k,s}\left(x_q\right)\left(\sum_{r=s}^{N}a_rT_{L,r}^{*,s}\left(x_q\right)\right)\left(\sum_{r=k}^{N}a_rT_{L,r}^{*,k}\left(x_q\right)\right)=f\left(x_q\right),$$

where x_p are the roots of $T_{L,m}^{*}(x)$. Further, now from Eq. (9) and putting it into the conditions in Eq. (5) we have k equations as follows:

(13)
$$\sum_{k=0}^{m-1}\left(a_{ik}\sum_{r=k}^{N}a_rT_{L,r}^{*,k}(a)+b_{ik}\sum_{r=0}^{N}a_rT_{L,r}^{*,k}(b)+c_{ik}\sum_{r=0}^{N}a_rT_{L,r}^{*,k}(c)\right)$$

(14)
$$=\alpha_i, i=0,1,\ldots,m-1.$$

Eq. (11), together with ks- equations from the conditions Eq. (12) give ($N+1$) non-linear algebraic equations (NAEs). By solving the NAEs, we obtain the unknown shifted Chebyshev coefficients $a_r, r=0,1,\ldots,N$. Consequently, using the calculated $y_N(x)$ we can have an estimate of $y(x)$ given in Eq. (4).

3.3. Error Analysis

Here we consider how the mention method is put together: we suppose that $y(x)$ is smooth on $[0,1]$ and $I_N(x)$ is the polynomial that interpolates y at the nodes $x_i, i=0,1,\ldots,n$ with x_i, being the Cbebyshev-Gauss node points. We then have [9, 10]

(15)
$$y(x)-I_N(x)=\frac{y^{(N+1)}(\lambda)}{(N+1)!}\prod_{i=0}^{N}\left(x-x_i\right), \lambda\in[0,1].$$

Also, we have

(16)
$$|y(x)-I_N(x)|\leq\frac{1}{2^{2N+1}}\left\|y^{(N+1)}(x)\right\|_{\infty}$$

The least-square norm is given as

$$\|y\|_2=\left(\int_a^b w(x)|y(x)|^2dx\right)^{1/2},$$

with $w(x)$ being a weight function that is nonnegative.
Theorem Suppose that the functions in Eq. (4) are real $(N+1)$-times continuously differentiable functions on the interval [0,1] and

$$y_N(x) = \sum_{r=0}^{N} a_r T^*_{L,r}(x),$$

are the SCPs expansion of the analytical solution.
Let

$$\overline{y_N}(x) = \sum_{r=0}^{N} \overline{a_r} T^*_{L,r}(x),$$

be the numerical result generated by the SCM method, $\exists \Re\ \alpha$ such that

(17) $$\|y(x) - y_N(x)\|_2 \leq \alpha \frac{1}{2^{N+1}} \left\| y^{(N+1)}(x) \right\|_\infty + \sqrt{\frac{3\pi}{8}} \|A - \bar{A}\|_2,$$

where

$$A = \left[\ a_0 \quad a_1 \cdots a_N\ \right] \text{ and } \bar{A} = [\overline{a_0 a_1} \cdots \overline{a_N}].$$

Proof:
Let $y_N(x)$ is real-valued polynomials of degree $\leq N$ and $y_N(x)$ is the best approximation of $y(x)$. We can write

$$\|y(x) - y_N(x)\|_2 \leq \|y(x) - \overline{y_N}(x)\|_2 + \|\overline{y_N}(x) - y_N(x)\|_2 .$$

Using Eq. (11), we obtain

$$\begin{aligned} \|y(x) - y_N(x)\|_2 &= \left(\int_0^1 |y(x) - y_N(x)|^2 dx \right)^{1/2} \\ &\leq \left(\int_0^1 \left[\frac{1}{2^{N+1}(N+1)!} \left\| y^{(N+1)}(x) \right\|_\infty \right]^2 dx \right)^{1/2} \\ &= \sqrt{L} \frac{1}{2^{2N+1}(N+1)!} \left(\left\| y^{(N+1)}(x) \right\|_\infty \right)^{N+1}, \end{aligned}$$

and we have

$$\left\| y(x) - \overline{y_N(x)} \right\|_2 = \left(\int_0^1 \left[\sum_{r=0}^{N} (a_r - \overline{a_r})\, T^*_r(x) \right]^2 dx \right)^{1/2}.$$

$$\leq \left(\int_0^1 \left[\sum_{r=0}^{N} (a_r - \overline{a_r})^2 \right] \left[\sum_{r=0}^{N} |T^*_r(x)|^2 \right] dx \right)^{1/2}$$

$$= \left[\sum_{r=0}^{N} (a_r - \overline{a_r})^2 \right]^{1/2} \left(\sum_{r=0}^{N} \int_0^1 |T^*_r(x)|^2 dx \right)^{1/2} = \sqrt{\frac{3\pi}{8}} \|A - \bar{A}\|.$$

Furthermore, we can easily verify the accuracy of the method. Since the truncated Chebyshev series(3) is an approximate solutions of Eq. (1), when we substitute the function $y_j^N(x), j = 0, 1, \ldots, m$ and its first derivatives are substituted in Eq. (1) he equation must be approximately satisfied; that is, for $x_i \in [0, 1], i = 0, 1, 2, \ldots$

$$\left| \sum_{k=0}^{m} \sum_{s=0}^{n} Q_{k,s}(x_i)\, y^s(x_i)\, y^{(k)}(x_i) + \sum_{k=1}^{m} \sum_{s=1}^{m} P_{k,s}(x_i)\, y^{(s)}(x_i)\, y^{(k)}(x_i) - f(x_i) \right| \cong 0,$$

therefore, the estimated error can be given by the function

$$E_N(x) = \sum_{k=0}^{m}\sum_{s=0}^{n} Q_{k,s}(x)y^s(x)y^{(k)}(x) + \sum_{k=1}^{m}\sum_{s=1}^{m} P_{k,s}(x)y^{(s)}(x)y^{(k)}(x) - f(x).$$

4. Interpretation and Description of the results

In this section, we examine both homogeneous and non-homogeneous Painlevé equations to demonstrate the effectiveness and versatility of the proposed method.

4.1. Example 1

Consider the first order initial value problem

$$\left(1+\sqrt{2}\right)y'(x) + xy(x) = 0, \quad y(x) = 0. \tag{18}$$

The Exact solution of eq.(18) is given as

$$y(x) = e^{\frac{x^2}{2} - \frac{x^2}{\sqrt{2}}}. \tag{19}$$

By considering the shifted Chebyshev polynomial in equation (6) and (7), for $N = 6$, we obtain

$$\begin{aligned} y_N x &= a_0 + (-1+2x)a_1 + 2\left(\frac{1}{2} - 4x + 4x^2\right)a_2 + 3\left(-\frac{1}{3} + 6x - 16x^2 + \frac{32x^3}{3}\right)a_3 \\ &+4\left(\frac{1}{4} - 8x + 40x^2 - 64x^3 + 32x^4\right)a_4 + 5\left(-\frac{1}{5} + 10x - 80x^2 + 224x^3 - 256x^4 + \frac{512x^5}{5}\right)a_5 \\ &+6\left(\frac{1}{6} - 12x + 140x^2 - \frac{1792x^3}{3} + 1152x^4 - 1024x^5 + \frac{1024x^6}{3}\right)a_6. \end{aligned} \tag{20}$$

By considering the initial equation, we obtain

$$y(0) = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 = 1. \tag{21}$$

Expanding Eq.(20) leads to

$$\begin{aligned} R &= xa_0 + 2a_1 + 2\sqrt{2}a_1 - xa_1 + 2x^2a_1 - 8a_2 - 8\sqrt{2}a_2 + 17xa_2 + 16\sqrt{2}xa_2 - 8x^2a_2 + 8x^3a_2 \\ &+18a_3 + 18\sqrt{2}a_3 - 97xa_3 - 96\sqrt{2}xa_3 + 114x^2a_3 + 96\sqrt{2}x^2a_3 - 48x^3a_3 + 32x^4a_3 - 32a_4 \\ &-32\sqrt{2}a_4 + 321xa_4 + 320\sqrt{2}xa_4 - 800x^2a_4 - 768\sqrt{2}x^2a_4 + 672x^3a_4 + 512\sqrt{2}x^3a_4 - 256x^4a_4 \\ &+128x^5a_4 + 50a_5 + 50\sqrt{2}a_5 - 801xa_5 - 800\sqrt{2}xa_5 + 3410x^2a_5 + 3360\sqrt{2}x^2a_5 - 5520x^3a_5 \\ &-5120\sqrt{2}x^3a_5 + 3680x^4a_5 + 2560\sqrt{2}x^4a_5 - 1280x^5a_5 + 512x^6a_5 - 72a_6 - 72\sqrt{2}a_6 + 1681xa_6 \\ &+1680\sqrt{2}xa_6 - 10824x^2a_6 - 10752\sqrt{2}x^2a_6 + 28488x^3a_6 + 27648\sqrt{2}x^3a_6 - 34304x^4a_6 - \\ &30720\sqrt{2}x^4a_6 + 19200x^5a_6 + 12288\sqrt{2}x^5a_6 - 6144x^6a_6 + 2048x^7a_6 = 0. \end{aligned} \tag{22}$$

The roots of the shifted Chebyshev polynomial for $N = 6$ is given as

(23)
$$x=\frac{1}{2}-\frac{1}{2}\text{Cos}\left[\frac{90}{7}\right];\quad x=\frac{1}{2}-\frac{1}{2}\text{Cos}\left[\frac{270}{7}\right];\quad x=\frac{1}{2}-\frac{1}{2}\text{Cos}\left[\frac{450}{7}\right];\quad x=\frac{1}{2};$$
$$x=\frac{1}{2}+\frac{1}{2}\text{Cos}\left[\frac{450}{7}\right];\quad x=\frac{1}{2}+\frac{1}{2}\text{Cos}\left[\frac{270}{7}\right].$$

Substituting Eq.(23) in Eq.(22), we obtain seven nonlinear algebraic equation with seven unknowns.

(24)
$$0.0209886a_0+4.80832a_1-18.4854a_2+38.68a_3-61.8371a_4+83.6317a_5-99.5317a_6=0.$$
$$0.178473a_0+4.71366a_1-12.4506a_2+9.62892a_3+8.42484a_4-29.5432a_5+32.8896a_6=0.$$
$$0.441666a_0+4.7769a_1-2.68295a_2-13.5448a_3+9.16209a_4+20.0266a_5-19.1649a_6=0.$$
$$0.5a_0+4.82843a_1-0.5a_2-14.4853a_3+0.5a_4+24.1421a_5-0.5a_6=0.$$
$$0.558334a_0+4.89357a_1+1.71017a_2-13.8885a_3-8.2695a_4+20.5785a_5+18.4011a_6=0.$$
$$0.821527a_0+5.35671a_1+12.2777a_2+8.76341a_3-9.365a_4-29.8869a_5-32.3914a_6=0.$$
$$a_0-a_1+a_2-a_3+a_4-a_5+a_6=1.$$

Solving nonlinear system of Eq.(24) leads to

(25)
$$a_0=0.92788\ \ a_1=-0.09471\ \ a_2=-0.02153\ \ a_3=0.00119 a_4=0.00012$$
$$a_5=-7.59957\ \ a_6=-5.41075.$$

And the approximate solution is given as

(26)
$$y_N=1-5.281078521521959 10^{-7}x-0.207091x^2-0.000115016x^3$$
$$+0.0218123x^4-0.000566612x^5-0.00110812x^6.$$

The figures below depict the graph of exact with approximate solutions, and error for $N=6$.

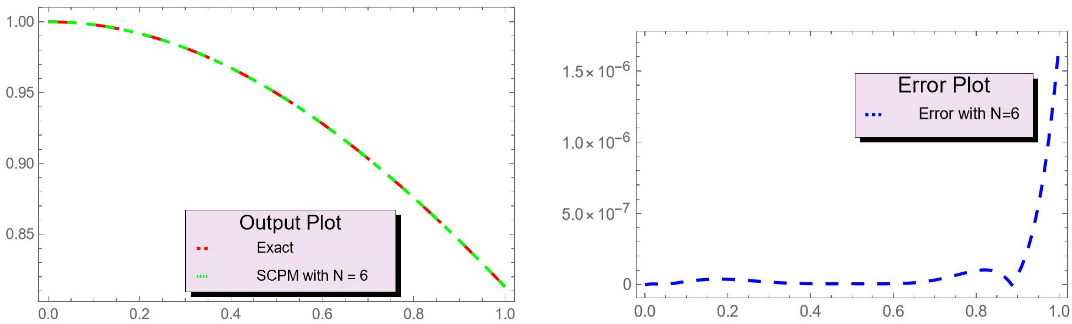

Figure 1: Plot of exact with approximate solutions, and error for $N=6$.

Table 1: Error of Example 1 for $N = 6$.

	Exact solution	Numerical solution	Absolute errors
0	1	1.	1.11022×10^{-16}
0.1	0.99793	0.99793	2.23859×10^{-8}
0.2	0.99175	0.99175	3.74590×10^{-8}
0.3	0.98153	0.98153	1.84683×10^{-8}
0.4	0.96740	0.96740	6.49594×10^{-9}
0.5	0.94954	0.94954	5.95489×10^{-9}
0.5	0.94954	0.94954	5.95489×10^{-9}
0.6	0.92815	0.92815	6.55413×10^{-9}
0.7	0.90349	0.90349	3.26183×10^{-8}
0.8	0.87586	0.87586	9.86706×10^{-8}
0.9	0.84556	0.84556	6.74423×10^{-8}
1	$e^{-\frac{1}{2(1+\sqrt{2})}}$	0.81293	1.68531×10^{-6} .

4.2. Example 2

Consider the first order initial value problem

$$y''(x)+\frac{1}{\sqrt{2}}y'(x)+e^{x}y(x)=0, \quad y(0)=1,\ y'(0)=0.9. \tag{27}$$

The Exact solution of eq.(27) is given as

$$y_N(x)=-\text{“3.72463”}e^{-\frac{x}{2\sqrt{2}}}\left(\text{“1.”BesselJ}\left[-\frac{1}{\sqrt{2}},2\sqrt{e^{x}}\right]+\text{“0.259053”BesselJ}\left[\frac{1}{\sqrt{2}},2\sqrt{e^{x}}\right]\right). \tag{28}$$

By considering the shifted Chebyshev polynomial in equation (6) and (7), for $N = 3$, we obtain

$$Y=a_0+(-1+2x)a_1+2\left(\frac{1}{2}-4x+4x^2\right)a_2+3\left(-\frac{1}{3}+6x-16x^2+\frac{32x^3}{3}\right)a_3=0. \tag{29}$$

By considering the initial equation in (27) , we obtain

$$y(0)=\ a_0-a_1+a_2-a_3=1 \quad and y'(0)=\ 2a_1-8a_2+18a_3=0.9. \tag{30}$$

Expanding Eq.(20) leads to

$$\begin{aligned}R=&e^{x}a_0+\sqrt{2}a_1-e^{x}a_1+2e^{x}xa_1+16a_2-4\sqrt{2}a_2+e^{x}a_2+8\sqrt{2}xa_2-8e^{x}xa_2+8e^{x}x^2a_2-96a_3\\&+9\sqrt{2}a_3-e^{x}a_3+192xa_3-48\sqrt{2}xa_3+18e^{x}xa_3+48\sqrt{2}x^2a_3-48e^{x}x^2a_3+32e^{x}x^3a_3=0.\end{aligned} \tag{31}$$

The roots of the shifted Chebyshev polynomial for $N = 3$ is given as

$$x=\frac{1}{4}\left(2-\sqrt{2}\right);\quad x=\frac{1}{4}\left(2+\sqrt{2}\right): \tag{32}$$

Substituting Eq.(32) in Eq.(31), we obtain four nonlinear algebraic equation with four unknowns

$$\begin{aligned}1.15771a_0+0.595587a_1+12.a_2-62.821a_3=0.\\2.34798a_0+3.07448a_1+20.a_2+70.4646a_3=0.\\a_0-a_1+a_2-a_3=1\\2a_1-8a_2+18a_3=0.9.\end{aligned} \tag{33}$$

Solving Eq.(33) leads to

(34) $$a_0 = 1.11008, \quad a_1 = -0.00749691, \quad a_2 = -0.12014, \quad a_3 = -0.00256276.$$

And the approximate solution is given as

(35) $$y_N = 1 + 0.9x - 0.838111x^2 - 0.0820084x^3.$$

By considering Example 2 for $N = 6$ and using the shifted Chebyshev polynomial in equation (3) and (4), for $N = 6$, we obtain

(36) $$\begin{aligned} Y = a_0 + (-1+2x)a_1 + 2\left(\frac{1}{2} - 4x + 4x^2\right)a_2 + 3\left(-\frac{1}{3} + 6x - 16x^2 + \frac{32x^3}{3}\right)a_3 \\ +4\left(\frac{1}{4} - 8x + 40x^2 - 64x^3 + 32x^4\right)a_4 + 5\left(-\frac{1}{5} + 10x - 80x^2 + 224x^3 - 256x^4 + \frac{512x^5}{5}\right)a_5 \\ +6\left(\frac{1}{6} - 12x + 140x^2 - \frac{1792x^3}{3} + 1152x^4 - 1024x^5 + \frac{1024x^6}{3}\right)a_6. \end{aligned}$$

With the initial equation, we obtain

(37)
$$y(0) = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 = 1, \;\; y'(0) = 2a_1 - 8a_2 + 18a_3 - 32a_4 + 50a_5 - 72a_6 = 0.9.$$

Expanding Eq.(36) leads to

(38) $$\begin{aligned} Y = e^x a_0 + \sqrt{2}a_1 - e^x a_1 + 2e^x x a_1 + 16a_2 - 4\sqrt{2}a_2 + e^x a_2 + 8\sqrt{2}xa_2 - \\ 8e^x x a_2 + 8e^x x^2 a_2 - 96a_3 + 9\sqrt{2}a_3 - e^x a_3 + 192xa_3 - 48\sqrt{2}xa_3 + 18e^x x a_3 + \\ 48\sqrt{2}x^2 a_3 - 48e^x x^2 a_3 + 32e^x x^3 a_3 + 320a_4 - 16\sqrt{2}a_4 + e^x a_4 - \\ 1536xa_4 + 160\sqrt{2}xa_4 - 32e^x x a_4 + 1536x^2 a_4 - 384\sqrt{2}x^2 a_4 + 160e^x x^2 a_4 + \\ 256\sqrt{2}x^3 a_4 - 256e^x x^3 a_4 + 128e^x x^4 a_4 - 800a_5 + 25\sqrt{2}a_5 - e^x a_5 + \\ 6720xa_5 - 400\sqrt{2}xa_5 + 50e^x x a_5 - 15360x^2 a_5 + 1680\sqrt{2}x^2 a_5 - 400e^x x^2 a_5 + \\ 10240x^3 a_5 - 2560\sqrt{2}x^3 a_5 + 1120e^x x^3 a_5 + 1280\sqrt{2}x^4 a_5 - \\ 1280e^x x^4 a_5 + 512e^x x^5 a_5 + 1680a_6 - 36\sqrt{2}a_6 + e^x a_6 - 21504xa_6 + \\ 840\sqrt{2}xa_6 - 72e^x x a_6 + 82944x^2 a_6 - 5376\sqrt{2}x^2 a_6 + 840e^x x^2 a_6 - \\ 122880x^3 a_6 + 13824\sqrt{2}x^3 a_6 - 3584e^x x^3 a_6 + 61440x^4 a_6 - 15360\sqrt{2}x^4 a_6 + \\ 6912e^x x^4 a_6 + 6144\sqrt{2}x^5 a_6 - 6144e^x x^5 a_6 + 2048e^x x^6 a_6 = 0. \end{aligned}$$

The roots of the shifted Chebyshev polynomial for $N = 6$ is given as

(39)
$$x = \frac{1}{2};\; x = \frac{1}{4}\left(2 - \sqrt{\frac{1}{2}\left(5 - \sqrt{5}\right)}\right);\; x = \frac{1}{4}\left(2 + \sqrt{\frac{1}{2}\left(5 - \sqrt{5}\right)}\right);\; x = \frac{1}{4}\left(2 - \sqrt{\frac{1}{2}\left(5 + \sqrt{5}\right)}\right).$$

Substituting Eq.(39) in Eq.(38), we obtain seven nonlinear algebraic equation with seven unknowns

(40)
$$\begin{aligned}1.64872a_0+1.41421a_1+14.3513a_2-4.24264a_3-62.3513a_4+7.07107a_5+142.351a_6=0.\\ 1.22889a_0+0.691893a_1+12.2952a_2-53.6381a_3+71.7845a_4+13.4608a_5-186.494a_6=0.\\ 2.21199a_0+2.71439a_1+18.6415a_2+55.9442a_3+62.7693a_4-30.9415a_5-198.029a_6=0.\\ 1.02477a_0+0.439596a_1+11.4491a_2-80.7964a_3+266.238a_4-621.716a_5+1175.22a_6=0.\\ 2.65257a_0+3.93696a_1+23.526a_2+103.968a_3+301.561a_4+667.481a_5+1226.95a_6=0.\\ a_0-a_1+a_2-a_3+a_4-a_5+a_6=1\\ 2a_1-8a_2+18a_3-32a_4+50a_5-72a_6=0.9.\end{aligned}$$

Solving Eq.(40) leads to

(41)
$$\begin{aligned}a_0=1.10497 \quad a_1=-0.01569 \quad a_2=-0.12353a_3=-0.00236\\ a_4=0.00062 \quad a_5=0.00012 \quad a_6=8.53599.\end{aligned}$$

And the approximate solution is given as

(42) $$y_N(x)=1+0.9x-0.818148x^2-0.124678x^3-0.0227134x^4+0.0121948x^5+0.0174817x^6.$$

The figures below depict the comparison between the graph of exact with approximate solutions, and error for $N=3$ and that of $N=6$.

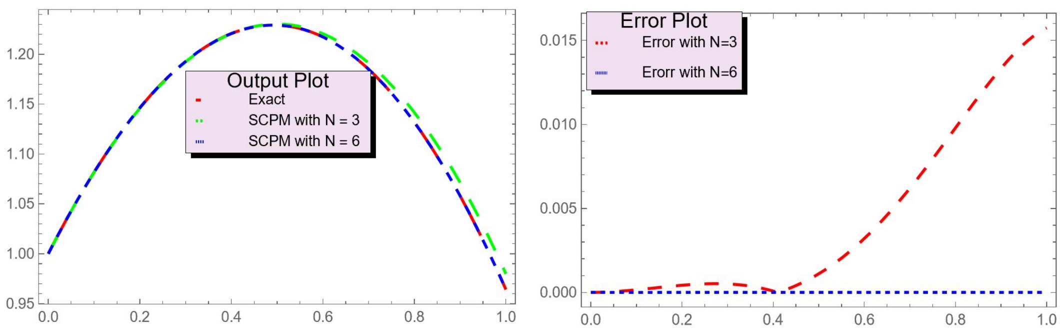

Figure 2: Plot of Example 2 of exact with approximate solutions, and error for $N=3$ and that of $N=6$

The Exact solution is

$$y=-3.72463e^{-\frac{x}{2\sqrt{2}}}\left(1.\text{BesselJ}\left[-\frac{1}{\sqrt{2}},2\sqrt{e^x}\right]+0.259053\text{BesselJ}\left[\frac{1}{\sqrt{2}},2\sqrt{e^x}\right]\right).$$

The approximate solution when $N=3$

$$y_N(x)=1+0.9x-0.838111x^2-0.0820084x^3.$$

The approximate solution when $N=6$

$$y_N(x)=1+0.9x-0.818148x^2-0.124678x^3-0.0227134x^4+0.0121948x^5+0.0174817x^6.$$

Table 2: Error of Example 2 for $N=3$ and $N=6$.

	Exact solution	Numerical solution N = 3	Numerical solution N = 6	Absolute errors N = 3	Absolute errors N = 6
0	1.	1.	1.	1.11022×10^{-16}	2.22044×10^{-16}
0.1	1.08169	1.08154	1.08169	0.00015	4.44788×10^{-8}
0.2	1.14625	1.14582	1.14625	0.00042	7.41713×10^{-7}
0.3	1.19286	1.19236	1.19286	0.00050	1.46181×10^{-6}
0.4	1.22073	1.22065	1.22073	0.00007	1.65312×10^{-6}
0.5	1.22911	1.23022	1.22911	0.00110	1.46583×10^{-6}
0.5	1.22911	1.23022	1.22911	0.00110	1.46583×10^{-6}
0.6	1.21736	1.22057	1.21736	0.00320	1.23582×10^{-6}
0.7	1.185	1.1912	1.185	0.00619	9.73220×10^{-7}
0.8	1.13183	1.14162	1.13183	0.00979	5.15671×10^{-7}
0.9	1.058	1.07135	1.058	0.01334	1.91159×10^{-7}
1	0.96413	0.97988	0.96413	0.01574	5.88723×10^{-7}

4.3. Example 3: Painleve Equation

Consider the first order initial value problem

(43) $$y'(x) - *y(x)^2 = x-1, y(0) = 0.5.$$

The exact solution of eq.(43) is given as

(44)
$$y = -(((\text{“0.5”} + \text{“0.866025”}i)\left((\text{“1.”} + \text{“0.”}i)\text{AiryAiPrime}\left[-(-1)^{1/3}(1-x)\right] + (\text{“0.52228”} + \text{“0.0809953”}i)\right.$$
$$\left.\left.\text{AiryBiPrime}\left[-(-1)^{1/3}(1-x)\right]\right)\right)\Big/$$
$$\left((\text{“1.”} + \text{“0.”}i)\text{AiryAi}\left[-(-1)^{1/3}(1-x)\right] + (\text{“0.52228”} + \text{“0.0809953”}i)\text{AiryBi}\left[-(-1)^{1/3}(1-x)\right]\right)\right).$$

By considering the shifted Chebyshev polynomial in equation (6) and (7), for $N=3$, we obtain

(45)
$$Y = a_0 + \left(-1 + \frac{1}{2}\left(2+\sqrt{3}\right)\right)a_1 + 2\left(-\frac{3}{2} - \sqrt{3} + \frac{1}{4}\left(2+\sqrt{3}\right)^2\right)a_2$$
$$+3\left(-\frac{1}{3} + \frac{3}{2}\left(2+\sqrt{3}\right) - \left(2+\sqrt{3}\right)^2 + \frac{1}{6}\left(2+\sqrt{3}\right)^3\right)a_3.$$

By considering the initial equation, we obtain

(46) $$y(0) = a_0 - a_1 + a_2 - a_3 = 0.5.$$

Expanding Eq.(45) leads to

(47) $$Y = 1 + 2a_1 + 4\sqrt{3}a_2 + 12a_3 = \frac{1}{2} + \frac{\sqrt{3}}{4} + a_0^2 + \sqrt{3}a_0a_1 + \frac{3a_1^2}{4} + a_0a_2 + \frac{1}{2}\sqrt{3}a_1a_2 + \frac{a_2^2}{4}.$$

The roots of the shifted Chebyshev polynomial for $N=3$ is given as

(48) $$x = \frac{1}{2}; \quad x = \frac{1}{4}\left(2-\sqrt{3}\right); \quad x = \frac{1}{4}\left(2+\sqrt{3}\right); \quad Z4 = a_0 - a_1 + a_2 - a_3 = 0.5.$$

Substituting Eq.(48) in Eq.(47), we obtain four nonlinear algebraic equation with four unknowns

(49)
$$1. + 2.a_1 + 6.9282a_2 + 12.a_3 = 0.933013 + a_0^2 + 1.73205a_0a_1 + 0.75a_1^2 + a_0a_2 + 0.866025a_1a_2 + 0.25a_2^2$$
$$1. + 2.a_1 + 6.9282a_2 + 12.a_3 = 0.933013 + a_0^2 + 1.73205a_0a_1 + 0.75a_1^2 + a_0a_2 + 0.866025a_1a_2 + 0.25a_2^2$$
$$1. + 2.a_1 + 6.9282a_2 + 12.a_3 = 0.933013 + a_0^2 + 1.73205a_0a_1 + 0.75a_1^2 + a_0a_2 + 0.866025a_1a_2 + 0.25a_2^2$$
$$a_0 - a_1 + a_2 - a_3 = 1.$$

Solving the nonlinear system of Eq.(49) leads to

$$a_0 = 0.23443 \quad a_1 = -0.22118 \quad a_2 = 0.04820 \quad a_3 = 0.00382. \tag{50}$$

And the approximate solution is given as

$$Y_N(x) = 0.5 - 0.759146x + 0.202027x^2 + 0.122402x^3. \tag{51}$$

The figures below depict the graph of exact with approximate solutions, exact solution, and error for $N = 6$.

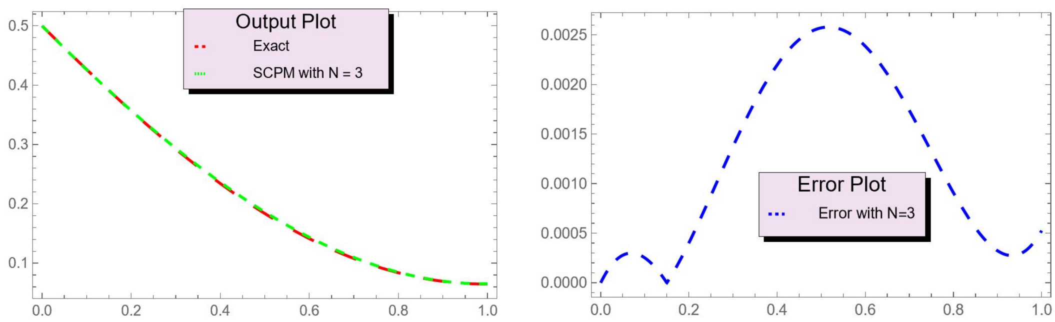

Figure 3: Plot of the approximate solutions, exact solution, and error for $N = 6$.

Table 3: Error of Example 3 for $N = 6$

	Exact solution	Numerical solution	Absolute errors
0	$0.5 + 2.77555 \times 10^{-17}\, i$	0.5	$2.77555 \times 10^{-17}\, i$
0.1	$0.42648 + 5.55111 \times 10^{-17}\, i$	0.42622	0.00025
0.2	$0.35683 + 8.32667 \times 10^{-17}\, i$	0.35723	0.00040
0.3	$0.29236 + 1.38777 \times 10^{-16}\, i$	0.29374	0.00137
0.4	$0.23430 + 1.66533 \times 10^{-16}\, i$	0.2365	0.00219
0.5	$0.18366 + 1.80411 \times 10^{-16}\, i$	0.18623	0.00257
0.5	$0.18366 + 1.80411 \times 10^{-16}\, i$	0.18623	0.00257
0.6	$0.14129 + 1.66533 \times 10^{-16}\, i$	0.14368	0.00238
0.7	$0.10783 + 2.49800 \times 10^{-16}\, i$	0.10957	0.00173
0.8	$0.08374 + 1.94289 \times 10^{-16}\, i$	0.08465	0.00090
0.9	$0.06931 + 2.42861 \times 10^{-16}\, i$	0.06964	0.00032
1	$0.06475 + 2.60208 \times 10^{-16}\, i$	0.06528	0.00052

5. Conclusion

The Shifted Chebyshev technique is an appropriate tool for the purpose of solving nonlinear ordinary differential equations. Its ability to solve complex problems in a simplified way using Chebyshev polynomials makes it efficient and accurate. It requires fewer computations compared to other methods, making it suitable for solving complex problems in many diverse scientific and engineering areas. The examples in this study show that the Shifted Chebyshev method can produce reliable and precise solutions. This makes it the best method for learners and researchers looking for a clear and effective way of solving nonlinear ordinary differential equations. The scope of the proposed method is limited to linear and nonlinear ODEs, it will an open problem to extend this idea to future researchers to investigates the PDES, System of ODEs, Fractional differential equations.

Authors' contribution

The authors has equal contribution in the framework of the manuscript.

Conflict of interests

There is no conflict of interest among any materials discussed in the article.

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