Eurasian J. Sci. Eng., 2025, 11(2), 272-277
https://doi.org/10.23918/eajse.v11i2p18
Published by Tishk International University
Available at: https://eajse.tiu.edu.iq/

RESEARCH ARTICLE Open Access

Applying Pourreza Transform to Solve Volterra Integro-Differential Equations of the 2nd Kind

Amal M. Wadi [1]*, and Nejmaddin A. Sulaiman [1]

[1] Department of Mathematics, Education College, Salahaddin University-Erbil, Iraq.

Article History
Received: 30.12.2024
Revised: 14.05.2025
Accepted: 17.08.2025
Published: 09.09.2025
Communicated by: Asst. Prof. Dr. Abubakar M. Ashir and Asst. Prof. Dr. Abdulnasir Isah
**Email address: amal.wadi@su.edu.krd*
**Corresponding-Author*

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Abstract: This paper's primary goal is to investigate and resolve the linear Volterra integro-differential equations of 2^{nd} kind with a convolution kernel. Here we present the HY transform called the P https://orcid.org/0000-0002-3307-9451ourreza transform for determining the resolution of the linear Volterra integro-differential equation of the 2^{nd} kind. One can represent in integral form a variety of subjects, including electromagnetic, radio physics, coagulation, meteorology, and population dynamics: biotechnology, radiation transfer, superfluidity, mining engineering, and acoustic engineering. The Pourreza transform has been applied to address several numerical problems; it can be converted into algebraic equations and solved in a few steps, demonstrating its versatility. Numerical problem results show how successful the Pourreza transform is in getting the accurate solution of the linear Volterra integro-differential equation of 2^{nd} kind.

Keywords: Convolution; Inverse Pourreza Transform; Pourreza Transform; Volterra Integro-Differential Equation.

1. Introduction

The linear Volterra integro-differential equation of the 2^{nd} kind (1) (LVI-DE OF 2^{nd} Kind) are given by (Vito Volterra, A mathematician and physicist from Italy who lived from 1860 to 1940) is well-known for his work in mathematical biology and integral equations. He was also one of the pioneers of functional analysis. As follows

$$\varphi^{(n)}(\omega) = \mathrm{H}(\omega) + \int_0^{\omega} \mathrm{K}(\omega, \xi)\, \varphi(\xi) d\xi$$

With

$$\varphi^{(m)}(0) = b_m \qquad , 0 < m < n - 1.$$

In which the unidentified function $\varphi(\xi)$, which will be decided, only shows up within the integral sign, whereas the derivative of script phi open paren omega, close paren usually takes place outside of the integral sign The Kernels $K(\omega, \xi)$, and the function $\mathrm{H}(\omega)$ provided functions with real values., b_m are constants that define the initial conditions. When Volterra studied a population growth model in the early 1900s, He established a unique class of integral equations by concentrating on genetic factors. The idea of integro-differential equations originated with this research. There exist formulations in which the integral and differential operators coexist within the same equation. There exist formulations in which the integral and differential operators coexist within the same equation. Many physical applications of the VI-DE have been found, such as the glass-forming process, heat transport, nanohydrodynamics, and dispersion processes generally. the coexistence of biological species with increasing and decreasing rates in neutron diffusion, and wind ripples in desert environments. Numerous disciplines, including biology, astronomy, engineering, physics, biotechnology, and radiology, provide extensive details on the use of these equations.

The function's Pourreza transform $\varphi(\omega)$ for all $\omega \geq 0$ is described as (2)

$$\mathrm{P}\{\varphi(\omega)\} = s\int_{0}^{+\infty} \varphi(\omega)\mathrm{e}^{-s^2\omega}\mathrm{d}\omega$$

where P is Pourreza transform operator and $\varphi(\omega)$ is a continuous piecewise function across all finite intervals in $\omega \in [0, +\infty)$ satisfying $|\varphi(\omega)| \leq M\mathrm{e}^{a\omega}$, $\exists M > 0$ for all $\omega \in [0, \infty)$, then $\mathrm{P}\{\varphi(\omega)\}$ (s) exists for all s > a. And the inversion transform is:

$$\mathrm{P}^{-1}\left\{s\int_{0}^{+\infty} \varphi(\omega)\mathrm{e}^{-s^2\omega}\mathrm{d}\omega\right\} = \mathrm{F}(\omega).$$

Integral transforms are crucial in figuring out the answers to complex issues in statistics, physics, medicine, agriculture, space research, and mathematics. The fact that these transforms provide the precise answer to the problem without requiring a lot of computational labour is their most alluring characteristic.

In recent years, several methods have been employed by certain researchers to resolve Volterra integral and integro-differential equations. Harish Nagar and Sonia Sharma applied the Complex Sadik Integral Transform for solving the LVI-DE OF 2^{nd} Kind (3). Dinesh T. and Prakash C. Thakur solved the LVI-E OF 2^{nd} Kind with a convolution-type kernel through the use of the Upadhyaya Transform (4). Zainab Rustam and Nejmaddin A. Sulaiman obtained the solutions of the system of LVI-DE OF 2^{nd}Kind by using various integral transformations (5, 6). Amal M.Wadi and Nejmaddin A. Sulaiman using W Transform for solving LVI-DE OF 2^{nd} Kind and the system of VI-DE OF 2^{nd} Kind in their work (7) . Sadiq A. Mehdi et al., employing the Complex SEE Integral-Transform to determine the precise answer to the LVI-DE OF 2^{nd} Kind(8). Hayriye and Haldun Alpaslan Peker solved the LVI-DE OF 2^{nd} Kind by Shehu Transform, The applications provided clearly demonstrate the Shehu transform's simplicity, efficiency, and great precision without requiring a lot of computational labour (9). Sudhanshu Aggarwal et al. discovered the first and second kind's analytical answers to VIE and VI-DE employing multiple integral transformations (Aboodh, Mohand, Sumudu, Mahgoub) (10, 11) (12, 13). A novel Integral transform was presented by Seyed Ahmad Pourreza Ahmadi et al. and employed in the resolution of higher-order linear ordinary Laguerre and Hermite differential equations(2). Integral transforms are essential for resolving practical issues. Integral and differential equations can both be simplified into readily solved algebraic equations by selecting the appropriate integral transformations. The objective of this work is to solve the 2^{nd} kind Volterra integro-differential equation in a few easily steps using the Pourreza transform.

2. Pourreza Transform to Solve LVI-DE OF 2^{nd} Kind

For the purpose of solving LVI-DE OF 2^{nd} Kind, we present in this section the Pourreza transform . We shall presume in this labor that the kernel $\mathrm{K}(\omega, \xi)$ of equation (2) is a difference kernel that differs in terms of expression $(\omega - \xi)$. The LVI-DE OF 2nd Kind can thus be expressed as

$$\varphi^{(n)}(\omega) = \mathrm{H}(\omega) + \int_{0}^{\omega} \mathrm{K}(\omega - \xi)\, \varphi(\xi)\mathrm{d}\xi \qquad (2)$$

Utilizing the convolution theorem of the Pourreza transform and the operator of the Pourreza transform on (2), we obtain

(3) $$P\{\varphi^{(n)}(\omega)\} = P\{\mathrm{H}(\omega)\} + \frac{1}{s}P\{\mathrm{K}(\omega)\}P\{\varphi(\omega)\}$$

By using the "Pourreza transforms of derivatives" characteristic to (3), we obtain

(4) $$s^{2n}P(\varphi(\omega)) - \sum_{k=0}^{n-1} s^{2(n-k)-1}\,\varphi^{(k)}(0) = P\{\mathrm{H}(\omega)\} + \frac{1}{s}P\{\mathrm{K}(\omega)\}P\{\varphi(\omega)\}$$

After simplification of equation (4), we have the values of $P\{\mathrm{H}(\omega)\}, P\{\mathrm{K}(\omega)\}$. We obtain the necessary value of $\varphi(\omega)$by performing the inverse Pourreza transform on these data.

Table 1: The Pourreza Transform's Properties

S.N	Property Name	Formulary Form
1.	The ability to linearize	$P\{c\varphi_1(\omega) + d\varphi_2(\omega)\} = cP\{\varphi_1(\omega)\} + dP\{\varphi_2(\omega)\}$
2.	First Derivative	$P\{\varphi'(\omega)\} = s^2P(\varphi(\omega)) - s\,\varphi(0)$
3.	Second Derivative	$P\{\varphi''(\omega)\} = s^4P(\varphi(\omega)) - s^3\,F(0) - \mathrm{s}\,\varphi'(0)$
4.	nth Derivative	$P\{\varphi^{(\mathrm{n})}(\omega)\} = s^{2n}P(\varphi(\omega)) - \sum_{k=0}^{n-1} s^{2(n-k)-1}\,\varphi^{(k)}(0),\ \ n \geq 1$
5.	Convolution	$P\{\varphi_1(\omega) * \varphi_2(\omega)\} = \frac{1}{s}\,P\{\varphi_1(\omega)\} * P\{\varphi_2(\omega)\}$

Table 2: Useful Functions Transformed by Pourreza

S.N	$\varphi(\omega)$	$P\{\varphi(\omega)\} = \varphi(s)$
1.	1	$\frac{1}{s}$
2.	ω	$\frac{1}{s^3}$
3.	ω^2	$\frac{2!}{s^5}$
4.	$\mathrm{a}\omega^n$	$\frac{a.n!}{s^{2n+1}}$, a constant
5.	ω^n	$\frac{\Gamma(n+1)}{s^{2n+1}}$, n > -1

6.	$e^{a\omega}$	$\frac{s}{s^2-a}$
7.	$sina\omega$	$\frac{a\,s}{s^4+a^2}$
8.	$cosa\omega$	$\frac{s^3}{s^4+a^2}$
9.	$sinha\omega$	$\frac{a\,s}{s^4-a^2}$
10.	$cosha\omega$	$\frac{s^3}{s^4-a^2}$

Table 3: Invers of Pourreza Transform

S.N	$\varphi(s)$	$\varphi(\omega)=P^{-1}\{\varphi(s)\}$
1.	$\frac{1}{s}$	1
2.	$\frac{1}{s^3}$	ω
3.	$\frac{2!}{s^5}$	ω^2
4.	$\frac{a.n!}{s^{2n+1}}$, a constant	$a\omega^n$
5.	$\frac{\Gamma(n+1)}{s^{2n+1}}$, n > -1	ω^n
6.	$\frac{s}{s^2-a}$	$e^{a\omega}$
7.	$\frac{a\,s}{s^4+a^2}$	$sina\omega$
8.	$\frac{s^3}{s^4+a^2}$	$cosa\omega$
9.	$\frac{a\,s}{s^4-a^2}$	$sinha\omega$
10.	$\frac{s^3}{s^4-a^2}$	$cosha\omega$

3. Numerical Problems

This section presents several applications that illustrate the efficiency of the Pourreza transform in resolving LVI-DE OF 2^{nd}Kind.

Problem 1: Consider the LVI-DE OF 2^{nd} Kind.

(5) $$\varphi'(\omega) = 2 - \frac{1}{4}\omega^2 + \frac{1}{4}\int_0^{\omega} \varphi(\xi)d\xi$$

with $\varphi(0) = 0$

Operating Pourreza transform on equation (5)

(6) $$P\{\varphi'(\omega)\} = P\{2\} - P\{\frac{1}{4}\omega^2\} + P\{\frac{1}{4}\int_0^{\omega} \varphi(\xi)d\xi\}$$

By applying the Pourreza transform's convolution theorem on equation (6) , we get

(7) $$P\{\varphi'(\omega)\} = P\{2\} - P\left\{\frac{1}{4}\omega^2\right\} + \frac{1}{4s}P\{1\}P\{\varphi(\omega)\}$$

Pourreza transforms of derivatives are a property that can be used in equation (7), we have

(8) $$s^2 P\{\varphi(\omega)\} - s\varphi(0) = \frac{2}{s} - \frac{1}{2s^5} + \frac{1}{4s^2}P\{\varphi(\omega)$$

After simplification of equation (8) and operating inverse Pourreza transforms, we get the solution,

$$\varphi(\omega) = 2\omega.$$

Problem 2: Consider the LVI-DE OF 2^{nd} Kind.

(9) $$\varphi''(\omega) = -1 + \int_0^{\omega} (\omega - \xi)\, \varphi(\xi)d\xi$$

with $\varphi(0) = 1\,, \varphi'(0) = 0$

By transforming both sides of (9), using the Pourreza transform, we have

(10) $$P\{\varphi''(\omega)\} = P\{-1\} + P\{\int_0^{\omega} (\omega - \xi)\, \varphi(\xi)d\xi\}$$

and using the convolution theorem of the Pourreza transform on equation (10), we get

(11) $$P\{\varphi''(\omega)\} = P\{-1\} + \frac{1}{s}P\{\omega\}P\{\varphi(\omega)\}$$

Using the property "Pourreza transforms" on equation (11), we have

(12) $$s^4 P\{\varphi(\omega)\} - s^3\varphi(0) - s\varphi'(0) = -\frac{1}{s} + \frac{1}{s^4}P\{\varphi(\omega)\}$$

After simplification of equation (12) and operating inverse Pourreza transforms, we get the solution

$$\varphi(\omega) = \cos(\omega).$$

Problem 3: Consider the LVI-DE OF 2^{nd} Kind.

(13) $$\varphi''(\omega) = 1 + \omega + \int_0^{\omega} (\omega - \xi)\, \varphi(\xi)d\xi$$

with $\varphi(0) = 1\,, \varphi'(0) = 1$

When we apply the Pourreza transform to each of the two sides of (13), we get

(14) $$P\{\varphi''(\omega)\} = P\{1\} + P\{\omega\} + P\{\int_0^{\omega} (\omega - \xi)\, \varphi(\xi)d\xi\}$$

and using the convolution theorem of the Pourreza transform on equation (14), we get

$$P\{\varphi''(\omega)\} = P\{1\} + P\{\omega\} + \frac{1}{s}P\{\omega\}P\{\varphi(\omega)\} \tag{15}$$

Using the property "Pourreza transforms" on equation (15), we have

$$s^4P\{\varphi(\omega)\} - s^3\varphi(0) - s\varphi'(0) = \frac{1}{s} + \frac{1}{s^3} + \frac{1}{s^4}P\{\varphi(\omega)\} \tag{16}$$

After simplification of equation (16) and operating inverse of Pourreza transforms, we get solution

$$\varphi(\omega) = e^{\omega}.$$

4. Conclusion

In this paper, we've been effective in tackling the Pourreza transform for solving LVI-DE OF 2^{nd} Kind, and by considering three numerical issues, we have fully outlined the process. The solutions to these problems demonstrate the great usefulness and effectiveness of the Pourreza transform in identifying the solution of the LVI-DE OF 2^{nd} Kind. The cases presented demonstrate that a precise solution was obtained with minimal computing resources and in a remarkably short period.

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