# An Approach for Description of Elastic Parameters of Cross-Anisotropic Saturated Soils

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**Abstract: The processes of deposition and consolidation in natural soils or compaction in fill materials will typically produce soils that are initially cross-anisotropic (also known as transversely isotropic or orthotropic) in terms of both small strain elastic behaviour and large strain plastic behaviour. For small strain elastic behaviour, five elastic parameters are required to fully describe cross-anisotropic soils such as two Young's moduli ($E_v$ in vertical direction and $E_h$ in horizontal direction), two Poisson's ratios ($v_{vh}$ horizontal strain due to vertical strain and $v_{hh}$ horizontal strain due to horizontal strain) and an independent shear modulus such as $G_{hv}$ (a shear wave horizontally transmitted with vertically polarised). These five elastic parameters have been expressed in different fashions in the literature. In this paper results from derivations showed that it is possible to express the elastic parameters for cross-anisotropic soils in a different way than that expressed in the literature using as a function of independent measurements of two shear moduli and two constrained moduli from three pairs of bender/extender elements BEEs fitted on a cross-anisotropic soil sample in a triaxial apparatus and measuring or assuming the value of $v_{hh}$ using combination of triaxial testing system and bender/extender element testing system,**

**Keywords*: Elastic Parameters, Cross-Anisotropic Soils, Bender/Extender Elements, Shear and Constrained Moduli**

## 1. Introduction

For a shear wave, the direction of motion of the soil particles (the wave polarisation) is perpendicular to the direction of wave transmission, as shown in Figure 1 (Clayton, 2011). Hence, different shear wave velocities can be measured, depending upon the direction of the wave transmission and the direction of the wave polarisation, e.g. $V_{svh}$, $V_{shv}$ and $V_{shh}$ (see Figure 2) (Hasan, 2016; Hasan & Wheeler, 2014), where the second subscript gives the wave transmission direction, the third subscript gives the wave polarisation direction and $v$ and $h$ represent vertical and horizontal respectively. For compression waves, the direction of particle motion (wave polarisation) is the same as the direction of wave transmission (see, Figure 1). By transmitting compression waves in vertical and horizontal directions, compression waves velocities $V_{pv}$ and $V_{ph}$ can be measured (see Figure 2).

The importance of anisotropy of very small strain behaviour has been investigated by many authors (Lee & Rowe, 2014; Simpson, Atkinson & Jovicic, 1996; Wongsaroj et al., 2004; Grammatikopoulou et al., 2014). They showed, using numerical analysis, that including anisotropy of $G$ during the prediction of deformations of tunnelling in stiff clays (such as London clay)
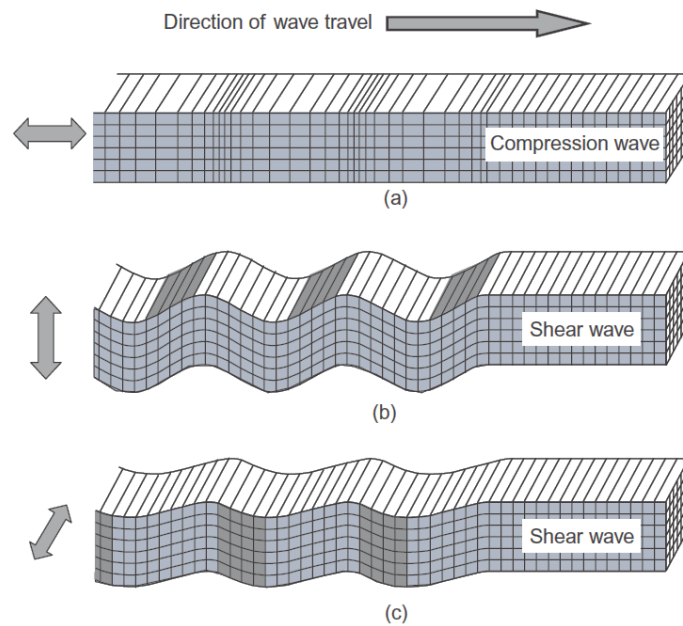
appeared to play a vital role.



Figure 1: Compression and shear wave travel: (a) Compression wave with horizontal transmission, $V_{ph}$ (b) Shear wave with horizontal transmission and vertical polarisation, $V_{shv}$ (c) Shear wave with horizontal transmission and horizontal polarisation, $V_{shh}$ (Clayton, 2011).
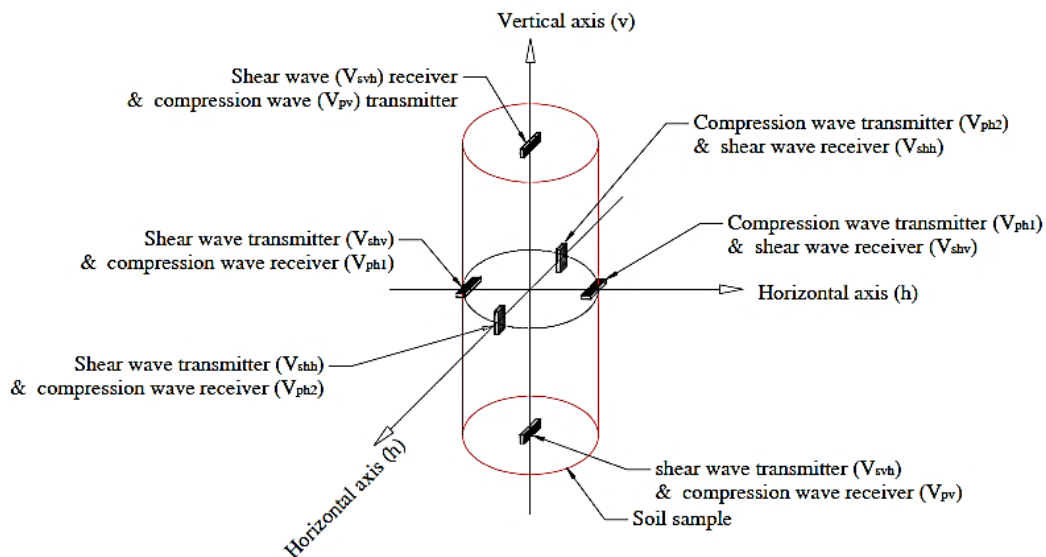


Figure 2: Multi-directional pairs of BEEs fitted on a triaxial soil sample (Hasan, 2016; Hasan & Wheeler, 2014).

This anisotropy of soil fabric can evolve during plastic straining, leading to changes in the anisotropy of mechanical behaviour. These changes of anisotropy caused by changes of soil fabric are therefore termed strain induced anisotropy (Jovicic & Coop, 1998). The 5 independent elastic constants of cross-anisotropic soils have been expressed using different techniques in the literature. For example,

two of them are presented here. Stokoe et al. (1995); Fioravante and Cappoferri (2001) showed that the 5 independent elastic constants of cross-anisotropic soils could be measured with bender/extender elements if an additional extender element was used to determine a constrained modulus *M* in an oblique direction. Alternatively, Pennington (1999) showed how all 5 independent constants could be determined by combining bender element testing (to measure $G_{hv}$ and $G_{hh}$) with local strain measurement on triaxial samples (to measure $E_v$, $E_h$ and $v_{vh}$). This does, however, have the drawback of combining two different types of measurement (at two different strain amplitudes). The main objective of this paper is to express the elastic parameters for cross-anisotropic saturated soils in a different fashion than those have been presented in the literature. The current expressions use a combination of triaxial testing system and bender/extender element testing system whereas other expressions employ, for example, either a triaxial testing system with local strain measurement on triaxial samples (Pennington, 1999) or a cubical calibration chamber with geophone systems (Stokoe et al., 1995). It means that additional equipment (i. e. bender/extender element testing system) in the geotechnical laboratory can be used to measure independent elastic constants and then express them.

## 2. Bender / Extender Element Testing

Bender/extender elements are piezoelectric transducers that can transmit and receive shear waves and compression waves in order to determine shear wave velocity $V_s$ and compression wave velocity $V_p$. These wave velocities can then be used to determine very small strain elastic values of shear modulus *G* and constrained modulus M as follows Biot (1956):

$$G = \rho V_s^2 \tag{1}$$

$$M = \rho V_p^2 \tag{2}$$

where $\rho$ is the bulk density of the soil. In all cases, the wave velocity *V* (i.e. $V_s$ or $V_p$) is determined from a measurement of travel time *t* and the known tip-to-tip distance $L_{tt}$ between transmitter and receiver elements (Viggiani & Atkinson, 1995):

$$V = \frac{L_{tt}}{t} \tag{3}$$

## 3. Elasticity Theory

### 3.1 Elastic Moduli of Isotropic Saturated Soils

At very small strains, the behaviour of saturated soils can be treated as elastic. If the soil is isotropic, the elastic behaviour can be represented by two independent elastic properties, which are normally selected either as Young's modulus *E* and Poisson's ratio v or as shear modulus *G* and bulk modulus *K* (Landau & Lifshitz, 1970), where:

$$G = \frac{E}{2(1+v)} \tag{4}$$

$$K = \frac{E}{3(1-2v)} \qquad (5)$$

For saturated soil, it can be helpful to choose to express the elastic properties in terms of $G$ and $K$ (rather than $E$ and $v$), because shear modulus $G$ should be the same for both drained and undrained behaviour, and $K$ is often considered as infinite for undrained behaviour. For linear elastic behaviour, $G$ and $K$ are constants, but soils often show non-linear elastic behaviour, with $G$ and $K$ varying with stress, strain or soil state.

Constrained modulus $M$ is the elastic modulus (applied normal stress increment divided by normal strain increment in the same direction) for a condition where strain is prevented in both perpendicular directions. For an isotropic elastic soil, $M$ can be expressed in terms of $E$ and $v$, or in terms of $G$ and $K$ (Landau & Lifshitz, 1970):

$$M = \frac{E(1-v)}{(1-2v)(1+v)} \qquad (6)$$

$$M = K + \frac{4}{3}G \qquad (7)$$

Equations (1) and (2) show that the shear wave velocity $V_s$ and compression wave velocity $V_p$ measured in bender/extender elements BEE tests depend upon $G$ (Equation 4) and $M$ (Equation 6 or 7), respectively.

## 3.2 Elastic Moduli of Anisotropic Saturated Soils

Love (1927) showed that thermodynamic considerations mean that the stiffness matrix (and compliance matrix) of an elastic material must be symmetric. This means that the most general form of linear anisotropic elastic behaviour involves 21 (rather than 36) independent elastic constants, for example see Graham and Houlsby (1983). For a cross-anisotropic elastic material, with the same properties in all horizontal directions but different properties in vertical directions, symmetry of the stiffness and compliance matrices implies that (Love, 1927):

$$\frac{v_{hv}}{E_h} = \frac{v_{vh}}{E_v} \qquad (8)$$

where $E_h$ and $E_v$ are the Young's moduli in horizontal and vertical directions respectively, $v_{hv}$ is the Poisson's ratio giving the ratio of vertical to horizontal strain increment caused by a uniaxial stress increment in the horizontal direction, and $v_{vh}$ is the Poisson's ratio giving the ratio of horizontal to vertical strain increment caused by a uniaxial stress increment in the vertical direction. Thermodynamic considerations also imply that for this cross-anisotropic material, the shear moduli $G_{vh}$, $G_{hv}$ and $G_{hh}$ are given by (Love, 1927):

$$G_{hv} = G_{vh} \qquad (9)$$

$$G_{hh} = \frac{E_h}{2(1+v_{hh})} \tag{10}$$

This means that, as shown by Graham & Houlsby (1983), the behaviour of a cross-anisotropic elastic material involves only 5 independent elastic constants, which can be taken as $E_v$, $E_h$, $v_{vh}$, $v_{hh}$ and $G_{vh}$. The stress-strain relations of this cross-anisotropic elastic material can then be expressed as (Love, 1927):

$$
\begin{bmatrix} \Delta\varepsilon_{xx} \\ \Delta\varepsilon_{yy} \\ \Delta\varepsilon_{zz} \\ \Delta\gamma_{xy} \\ \Delta\gamma_{yz} \\ \Delta\gamma_{zx} \end{bmatrix}
=
\begin{bmatrix}
\frac{1}{E_h} & \frac{-v_{hh}}{E_h} & \frac{-v_{vh}}{E_v} & & & \\
\frac{-v_{hh}}{E_h} & \frac{1}{E_h} & \frac{-v_{vh}}{E_v} & & & \\
\frac{-v_{vh}}{E_v} & \frac{-v_{vh}}{E_v} & \frac{1}{Ev} & & & \\
& & & \frac{1}{G_{hv}} & & \\
& & & & \frac{1}{G_{hv}} & \\
& & & & & \frac{2(1+v_{hh})}{E_h}
\end{bmatrix}
\begin{bmatrix} \Delta\sigma'_{xx} \\ \Delta\sigma'_{yy} \\ \Delta\sigma'_{zz} \\ \Delta\tau_{xy} \\ \Delta\tau_{yz} \\ \Delta\tau_{zx} \end{bmatrix}
\tag{11}
$$

For the conditions of the triaxial test, Equation (11) gives:

$$\Delta\varepsilon_v = \frac{1}{E_v}\Delta\sigma'_v - \frac{2v_{vh}}{E_v}\Delta\sigma'_h \tag{12}$$

$$\Delta\varepsilon_h = \left(\frac{1}{E_h} - \frac{v_{vh}}{E_h}\right)\Delta\sigma'_h - \frac{v_{vh}}{E_v}\Delta\sigma'_v \tag{13}$$

where $x$ and $y$ are horizontal directions, $z$ is the vertical direction, $\Delta\sigma'_{xx}$, $\Delta\sigma'_{yy}$ and $\Delta\sigma'_{zz}$ are normal stress increments, $\Delta\varepsilon_{xx}$, $\Delta\varepsilon_{yy}$ and $\Delta\varepsilon_{zz}$ are corresponding normal strain increments, $\Delta\tau_{yz}$, $\Delta\tau_{zx}$ and $\Delta\tau_{xy}$ are shear stress increments and $\Delta\gamma_{xy}$, $\Delta\gamma_{yz}$ and $\Delta\gamma_{zx}$ are corresponding shear strain increments.

## 4. Derivation of Expressions for $E_h$, $E_v$ and $V_{vh}$

### 4.1 Expressions for $M_v$

If a stress increment $\Delta\sigma'_v$ is applied in a vertical direction, to produce a corresponding strain

increment $\Delta\varepsilon_v$ in that direction, while strains are prevented in the horizontal direction (i.e. $\Delta\varepsilon_h=0$). Equation 13 can be re-arranged to give:

$$\Delta\sigma'_h = \frac{E_h}{E_v}\frac{v_{vh}}{(1-v_{vh})}\Delta\sigma'_v \tag{14}$$

By inserting Equation 14 into Equation 12:

$$\Delta\varepsilon_v = \left(\frac{1}{E_v} - \frac{2v_{vh}}{E_v}\frac{E_h}{E_v}\frac{v_{vh}}{(1-v_{hh})}\right)\Delta\sigma'_v \tag{15}$$

For this situation of no horizontal strain, the vertical constrained modulus $M_v$ is defined by (Powrie, 2014):

$$\Delta\varepsilon_v = \frac{\Delta\sigma'_v}{M_v} \tag{m(16)}$$

Comparing Equation 15 and 16:

$$M_v = \frac{E_v}{1-2v_{vh}\dfrac{E_h}{E_v}\dfrac{v_{vh}}{(1-v_{hh})}} \tag{17}$$

This simplifies to the standard result for the constrained modulus of an isotropic elastic material (see Equation 6) if $E_v = E_h = E$ and $v_{vh} = v_{hh} = v$.

### 4.2 Expressions for $M_h$

Consider a situation where a stress increment $\Delta\sigma'_{xx}$ is applied in one horizontal direction ($x$), to produce a corresponding strain increment ($\Delta\varepsilon_{xx}$) in that direction, while strains are prevented in the other horizontal direction ($\Delta\varepsilon_{yy}=0$) and in the vertical direction ($\Delta\varepsilon_{zz}=0$). Equation 11 now gives:

$$\Delta\varepsilon_x = \frac{1}{E_h}\Delta\sigma'_x - \frac{v_{vh}}{E_h}\Delta\sigma'_y - \frac{v_{vh}}{E_v}\Delta\sigma'_z \tag{18}$$

$$\Delta\varepsilon_y = -\frac{v_{hh}}{E_h}\Delta\sigma'_x + \frac{1}{E_h}\Delta\sigma'_y - \frac{v_{vh}}{E_v}\Delta\sigma'_z = 0 \tag{19}$$

$$\Delta\varepsilon_z = -\frac{v_{vh}}{E_v}\Delta\sigma'_x - \frac{v_{vh}}{E_v}\Delta\sigma'_y + \frac{1}{E_v}\Delta\sigma'_z = 0 \tag{20}$$

Solving the two simultaneous equations of Equations 19 and 20 for $\Delta\sigma'_y$ and $\Delta\sigma'_z$ gives:

$$\Delta\sigma_{y}^{'} = \left[\frac{v_{hh}E_v + v_{vh}^2 E_h}{E_v - v_{vh}^2 E_h}\right]\Delta\sigma_{x}^{'} \qquad (21)$$

$$\Delta\sigma_{z}^{'} = \left[\frac{(1+v_{hh})v_{vh}E_v}{E_v - v_{vh}^2 E_h}\right]\Delta\sigma_{x}^{'} \qquad (22)$$

Inserting for $\Delta\sigma'_y$ and $\Delta\sigma'_z$ from Equations 21 and 22 in Equation 18 and re-arranging:

$$\Delta\varepsilon_x = \left[\frac{(1-v_{hh}^2)E_v - 2v_{vh}^2(1+v_{hh})E_h}{E_h(E_v - v_{vh}^2 E_h)}\right]\Delta\sigma_{x}^{'} \qquad (23)$$

For this situation of zero strain in the $y$ (horizontal) and $z$ (vertical) directions, the horizontal constrained modulus $M_h$ is defined by:

$$\Delta\varepsilon_x = \frac{\Delta\sigma_{x}^{'}}{M_h} \qquad (24)$$

Comparing Equations 23 and 24:

$$M_h = \frac{E_h(E_v - 2v_{vh}^2 E_h)}{(1-v_{hh}^2)E_v - 2v_{vh}^2(1+v_{hh})E_h} \qquad (25)$$

This simplifies to the standard result for the constrained modulus of an isotropic elastic material (Equation 6) if $E_v = E_h = E$ and $v_{vh} = v_{hh} = v$.

## 4.3 Expressions for $E_h$, $E_v$ AND $v_{vh}$

One of the five independent elastic moduli of a cross-anisotropic soil ($G_{hv} = G_{hv}$) can be measured directly from one of the measurements provided by the standard arrangement of three BEE pairs. None of the other 4 independent elastic moduli of a cross-anisotropic soil ($E_h$, $E_v$, $v_{vh}$ and $v_{hh}$) can be determined from this standard arrangement of three BEE pairs, however if the value of one of them (say $v_{hh}$) is known or assumed, then it is possible to determine the values of the other three (say $E_h$, $E_v$ and $v_{vh}$) from the other three parameters measured by the standard arrangement of three BEE pairs ($G_{hh}$, $M_v$ and $M_h$).

Re-arranging Equation 10 gives:

$$E_h = 2(1+v_{hh})G_{hh} \qquad (26)$$

Inserting Equation 26 into Equation 17 and re-arranging:

$$(1-v_{hh})(M_h - E_v)E_v = 4v_{vh}^2(1+v_{hh})M_h G_{hh} \qquad (27)$$

Similarly, inserting Equation 26 into Equation 25:

$$[(1-v_{hh})M_h - 2G_{hh}]E_v = 4v_{vh}^2(1+v_{hh})(M_h - G_{hh})G_{hh} \qquad (28)$$

If $G_{hh}$, $M_v$ and $M_h$ are known, and $v_{hh}$ is either known or assumed, Equations 28 and 29 form two simultaneous equations in 2 unknowns ($E_v$ and $v_{vh}$). Solving:

$$E_v = \frac{(1+v_{hh})M_v G_{hh}}{(1-v_{hh})(M_h - G_{hh})} \qquad (29)$$

$$v_{vh} = \frac{1}{2(M_h - G_{hh})}\left[\frac{M_v\left((1-v_{hh})M_h - 2G_{hh}\right)}{(1-v_{hh})}\right]^{1/2} \qquad (30)$$

Equations 26, 29 and 30 provide expressions for the independent elastic moduli $E_h$, $E_v$ and $v_{vh}$ in terms of three of the moduli measured by the standard arrangement of three BEE pairs ($G_{hh}$, $M_v$ and $M_h$) and the final independent elastic modulus $v_{hh}$, the value of which must be either known independently or assumed. It is clear that BEE tests using the conventional arrangement of three pairs of BEEs (one transmitting vertically and two transmitting horizontally) provide only 4 independent measurements and hence cannot be used to determine all 5 independent elastic constants for a cross-anisotropic soil.

## 5. Conclusion

Based on the preceding derivations, it can be concluded that it is possible to use a combination of triaxial testing system and three pairs of bender/extender elements BEEs fitted on a cross-anisotropic soil sample, to express elastic parameters for cross-anisotropic soils in a different fashion than those have been expressed in the literature. Expressions for the independent elastic moduli $E_h$, $E_v$ and $v_{vh}$ are derived in terms of three of the moduli measured by the standard arrangement of three BEE pairs ($G_{hh}$, $M_v$ and $M_h$) and the final independent elastic modulus $v_{hh}$, the value of which must be either known independently or assumed.

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