

Approximate Solution of the Kersten-Krasil'shchik Coupled Kdv-Mkdv System via Reduced Differential Transform Method

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Abstract: In this paper, the approximate solution of the Kersten-Krasil'shchik coupled KdV-mKdV system is obtained by using the reduced differential transform method (RDTM). This system is regarded as a classical super-extension of the KdV equation. The obtained results are compared with the exact solutions to show the efficiency and reliability of the proposed method which can be extended to solve a large variety of nonlinear partial differential equations.

Keywords: Reduced Differential Transform Method, Kersten-Krasil'shchik Coupled KdV-mKdV System, Partial Differential Equations

1. Introduction

Nonlinear partial differential equations (NLPDEs) are widely used to describe complex phenomena arising in the real world. It occurs in a diversity of scientific fields such as optical fibers, fluid dynamics, acoustics, hydrodynamics, plasma physics, solid state physics. Recently, seeking analytical and approximate solutions of NLPDEs has become of great importance topic in the scientific milieu. Several groups of researchers have made magnificent exertions in this trend (Al-Amr & El-Ganaini, 2017; Al-Amr, 2015; Al-Sawoor & Al-Amr, 2012; Wazwaz, 2004). Among the NLPDEs, The Korteweg de-Vries (KdV) equation, which is a well-known NLPDE describes a diversity of physical phenomena, for instance, acoustic waves in a harmonic crystal and ion-acoustic waves in plasmas (Wazwaz, 2009). Numerous variations of this equation have been introduced heretofore by several scientists. Among these variations, the coupled KdV-mKdV system;

$$\begin{aligned} u_t + u_{xxx} - 6uu_x + 3v v_{xxx} + 3v_x v_{xx} - 3u_x v^2 + 6uv v_x &= 0, \\ v_t + v_{xxx} - 3v^2 v_x - 3uv_x + 3u_x v &= 0, \end{aligned} \quad (1)$$

which was recently introduced by Kersten and Krasil'shchik [12]. The coupled KdV-type equations usually arise as a description of shallow water waves [21]. In fact, setting $v = 0$ will turn system (1) into the well-known KdV equation;

$$u_t + u_{xxx} - 6uu_x = 0, \quad (2)$$

While for $u = 0$, it becomes the mKdV equation;

$$v_t + v_{xxx} - 3v^2 v_x = 0. \quad (3)$$

Which appears in multi-component plasmas, traffic flow, electric circuits, electrodynamics and elastic media (Wazwaz, 2009). In view of that, the system (1) can be considered as a sort coupling between (2) and (3).

The complete integrability of system (1) was proved by Kersten and Krasil'shchik (2000). Kalkanli, Sakovich and Yurdusen (2003) utilized the prolongation technique to obtain its Lax pair and investigated its integrability via the Painlevé test. Hon and Fan (2004) secured its travelling wave solutions including solitary wave solutions and doubly periodic wave solutions via an auxiliary equation method. Gepreel, Omran and Elegan (2011) employed the modified truncated expansion method to obtain its traveling wave solutions. Qin et al. (2012) derive its N -soliton solutions by means of the Bell polynomials. Rui and Qi (2016) employed the Hirota bilinear method to construct its quasi-periodic wave solution. To our knowledge, its approximate solution has not been investigated heretofore. To the best of our knowledge, the approximate solution of the Kersten-Krasil'shchik coupled KdV-mKdV system has not been investigated earlier by using any existing method in the literature. This was the motivation of this work. The RDTM is fruitfully employed to secure the approximate solution of the Kersten-Krasil'shchik coupled KdV-mKdV system.

The pattern of this paper is as follows: Section 2 handles the methodology of the utilized approach. In section 3, the approximate solution of the underlying system is obtained by using the RDTM. Finally, we draw some conclusions.

2. Overview of the Method

As one of the most effective solution methods, the RDTM was developed by Keskin and Oturanc in 2009. Since that, it has been widely used for handling different kinds of differential equations by many scientists (Al-Sawoor & Al-Amr, 2014; Saeed & Muhammad, 2016; Saeed & Mustafa, 2017). This method minimizes the volume of computations which make it easy to apply. It does not require additional mathematical operations. The closed form solution of RDTM is constructed as an infinite power series with straightforwardly computed terms. The more the number of computed terms, the more accurate a solution will be achieved.

To describe this method, let $u(x, t)$ be a function of two variables that could be expressed as $u(x, t) = f(x)g(t)$ where $f(x)$ and $g(x)$ are single-variable functions. According to the one-dimensional differential transform, we can write (Keskin & Oturanc, 2009);

$$u(x, t) = \left(\sum_{n=0}^{\infty} F(n) x^n \right) \left(\sum_{m=0}^{\infty} G(m) t^m \right) = \sum_{i=0}^{\infty} U_i(x) t^i, \quad (4)$$

where $U_i(x)$ is called the t -dimensional spectrum function of $u(x, t)$.

It is worthwhile to indicate that the notion of this method is followed from the power series expansion according to the following essential definitions (Keskin & Oturanc, 2009, 2010);

Definition 2.1: Suppose that $u(x, t)$ is an analytic and differentiated continuously with respect to x and t in the associated domain, then the transformed function $U_i(x)$, which is called the t -dimensional spectrum function, is defined by

$$U_i(x) = \frac{1}{i!} \left[\frac{\partial^i}{\partial t^i} u(x, t) \right]_{t=0}. \quad (5)$$

Definition 2.2: The differential inverse transform of the transformed function $U_i(x)$ can be defined as

$$u(x, t) = \sum_{i=0}^{\infty} U_i(x) t^i, \quad (6)$$

By inserting (5) into (6), we have

$$u(x, t) = \sum_{i=0}^{\infty} \frac{1}{i!} \left[\frac{\partial^i}{\partial t^i} u(x, t) \right]_{t=0} t^i. \quad (7)$$

The underlying transformations achieved by the RDTM are listed in Table 1. To explain the basic idea of the method, we consider the nonlinear partial differential equation (Keskin & Oturanc, 2009, 2010);

$$Lu(x, t) + Ru(x, t) + Nu(x, t) = g(x, t), \quad (8)$$

$$u(x, 0) = f(x), \quad (9)$$

where, $L = \frac{\partial}{\partial t}$, R, N are linear and nonlinear operators respectively and $g(x, t)$ is an inhomogeneous term.

Following the RDTM, the recursive formula of (8) could be easily found as:

$$(i+1)U_{i+1}(x, t) = G_i(x) - RU_i(x) - NU_i(x), \quad (10)$$

where $U_i(x)$, $RU_i(x)$, $NU_i(x)$ and $G_i(x)$ are the transformed functions of the original functions.

Table 1: The underlying transformations by using the RDTM

Function	Transformed Function
$h(x, t) = u(x, t) \pm v(x, t)$	$H_i(x) = U_i(x) \pm V_i(x)$
$h(x, t) = \alpha u(x, t)$	$H_i(x) = \alpha U_i(x)$ (α is a constant)
$h(x, t) = x^m t^n$	$H_i(x) = x^m \delta(i - n)$
$h(x, t) = x^m t^n u(x, t)$	$H_i(x) = x^m U_{i-n}(x)$
$h(x, t) = u(x, t)v(x, t)$	$H_i(x) = \sum_{k=0}^i V_k(x)U_{i-k}(x) = \sum_{k=0}^i U_k(x)V_{i-k}(x)$
$h(x, t) = \frac{\partial^i}{\partial t^i} u(x, t)$	$H_i(x) = (i+1) \cdots (i+k)U_{i+k}(x) = \frac{(i+k)!}{i!} U_{i+k}(x)$
$h(x, t) = \frac{\partial}{\partial x} u(x, t)$	$H_i(x) = \frac{\partial}{\partial x} U_i(x)$

The zeroth component could be acquired from the initial condition (9) as

$$U_0(x) = f(x), \quad (11)$$

Inserting (11) into (10) and performing iterative calculations, we obtain the following $U_i(x)$ values. Then, the n -terms approximate solution could be obtained by taking the inverse transformation of the sequence $\{U_i(x)\}_{i=0}^n$ as

$$\tilde{u}_n(x,t) = \sum_{i=0}^n U_i(x) t^i, \quad (12)$$

consequently, the analytical solution of (10) reads

$$u(x,t) = \lim_{n \rightarrow \infty} \tilde{u}_n(x,t). \quad (13)$$

3. Application

This section comprises the utilization of the RDTM to solve the coupled KdV-mKdV system given by (1) with initial conditions (Hon & Fan, 2004);

$$\begin{aligned} u(x,0) &= c - 2c \operatorname{sech}^2(\sqrt{c}x), \\ v(x,0) &= 2\sqrt{c} \operatorname{sech}(\sqrt{c}x), \end{aligned} \quad (14)$$

where c is a constant.

In accordance with the RDTM, Eqs. (1) could be transformed into the following iterative formula:

$$\begin{aligned} (i+1)U_{i+1}(x) &= -\frac{\partial^3}{\partial x^3} U_i(x) + 6A_i(x) - 3B_i(x) - 3C_i(x) + 3D_i(x) + 6E_i(x), \\ (i+1)V_{i+1}(x) &= -\frac{\partial^3}{\partial x^3} V_i(x) + 3F_i(x) + 3G_i(x) - 3H_i(x), \end{aligned} \quad (15)$$

where $U_i(x)$ and $V_i(x)$ are the transformed functions of $u(x,t)$ and $v(x,t)$ respectively. $A_i(x), B_i(x), C_i(x), D_i(x), E_i(x), F_i(x), G_i(x)$ and $H_i(x)$ are transformed forms of the nonlinear terms given by:

$$A_i(x) = \sum_{k=0}^i U_k(x) \frac{\partial}{\partial x} U_{i-k}(x), \quad (16)$$

$$B_i(x) = \sum_{k=0}^i V_k(x) \frac{\partial^3}{\partial x^3} V_{i-k}(x), \quad (17)$$

$$C_i(x) = \sum_{k=0}^i \frac{\partial}{\partial x} V_k(x) \frac{\partial^2}{\partial x^2} V_{i-k}(x), \quad (18)$$

$$D_i(x) = \sum_{k=0}^i \sum_{s=0}^k \frac{\partial}{\partial x} U_{i-k}(x) V_{k-s}(x) V_s(x), \quad (19)$$

$$E_i(x) = \sum_{k=0}^i \sum_{s=0}^k U_{i-k}(x) V_{k-s}(x) \frac{\partial}{\partial x} V_s(x), \quad (20)$$

$$F_i(x) = \sum_{k=0}^i \sum_{s=0}^k V_{i-k}(x) V_{k-s}(x) \frac{\partial}{\partial x} V_s(x), \quad (21)$$

$$G_i(x) = \sum_{k=0}^i U_k(x) \frac{\partial}{\partial x} V_{i-k}(x), \quad (22)$$

$$H_i(x) = \sum_{k=0}^i \frac{\partial}{\partial x} U_k(x) V_{i-k}(x). \quad (23)$$

Beginning with initial conditions (14) and by recursive steps, yields

$$\begin{aligned} U_0(x) &= c - 2c \operatorname{sech}^2(\sqrt{cx}), \\ U_1(x) &= \frac{8c^{5/2} \sinh(\sqrt{cx})}{\cosh(\sqrt{cx})^3}, \\ U_2(x) &= -\frac{8\left(2\cosh(\sqrt{cx})^2 - 3\right)c^4}{\cosh(\sqrt{cx})^4}, \\ U_3(x) &= \frac{64 \sinh(\sqrt{cx})c^{11/2}\left(\cosh(\sqrt{cx})^2 - 3\right)}{3 \cosh(\sqrt{cx})^5} \\ U_4(x) &= -\frac{32\left(2\cosh(\sqrt{cx})^4 - 15\cosh(\sqrt{cx})^2 + 15\right)c^7}{3 \cosh(\sqrt{cx})^6} \\ &\vdots \end{aligned} \quad (24)$$

and

$$\begin{aligned} V_0(x) &= 2\sqrt{c} \operatorname{sech}(\sqrt{cx}), \\ V_1(x) &= -\frac{4\sinh(\sqrt{cx})c^2}{\cosh(\sqrt{cx})^2}, \\ V_2(x) &= \frac{4c^{7/2}\left(\cosh(\sqrt{cx})^2 - 2\right)}{\cosh(\sqrt{cx})^3}, \\ V_3(x) &= -\frac{8\left(\cosh(\sqrt{cx})^2 - 6\right)c^5 \sinh(\sqrt{cx})}{3 \cosh(\sqrt{cx})^4} \\ V_4(x) &= \frac{4c^{13/2}\left(\cosh(\sqrt{cx})^4 - 20\cosh(\sqrt{cx})^2 + 24\right)}{3 \cosh(\sqrt{cx})^5} \\ &\vdots \end{aligned} \quad (25)$$

and so on, similarly, the rest of components could be secured with the aid of MAPLE. Now, the n -terms approximate solutions of the system could be acquired by taking the inverse transformation of the t -dimensional spectrum functions $\{U_i(x)\}_{i=0}^n$ and $\{V_i(x)\}_{i=0}^n$ as

$$\begin{aligned} \tilde{u}_n(x,t) = \sum_{i=0}^n U_i(x) t^i = c - 2c \operatorname{sech}^2(\sqrt{c}x) + \frac{8c^{5/2} \sinh(\sqrt{c}x)}{\cosh(\sqrt{c}x)^3} t \\ + \dots + \frac{1}{n!} \left[\frac{\partial^n}{\partial t^n} (c - 2c \operatorname{sech}^2(\sqrt{c}(x+2ct))) \right]_{t=0} t^n, \end{aligned} \quad (26)$$

$$\begin{aligned} \tilde{v}_n(x,t) = \sum_{i=0}^n V_i(x) t^i = 2\sqrt{c} \operatorname{sech}(\sqrt{c}x) + \frac{4 \sinh(\sqrt{c}x) c^2}{\cosh(\sqrt{c}x)^2} t \\ + \dots + \frac{1}{n!} \left[\frac{\partial^n}{\partial t^n} (2\sqrt{c} \operatorname{sech}(\sqrt{c}(x+2ct))) \right]_{t=0} t^n. \end{aligned} \quad (27)$$

Consequently, we can construct the exact solution of the system (1) as

$$\begin{aligned} u(x,t) = \lim_{n \rightarrow \infty} \tilde{u}_n(x,t) = c - 2c \operatorname{sech}^2(\sqrt{c}(x+2ct)), \\ v(x,t) = \lim_{n \rightarrow \infty} \tilde{v}_n(x,t) = 2\sqrt{c} \operatorname{sech}(\sqrt{c}(x+2ct)). \end{aligned} \quad (28)$$

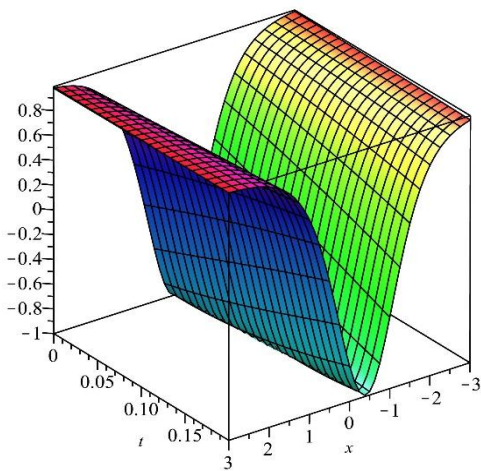


Fig. 1: Profile of the approximate solution $\tilde{u}_4(x,t)$ when $c=1$

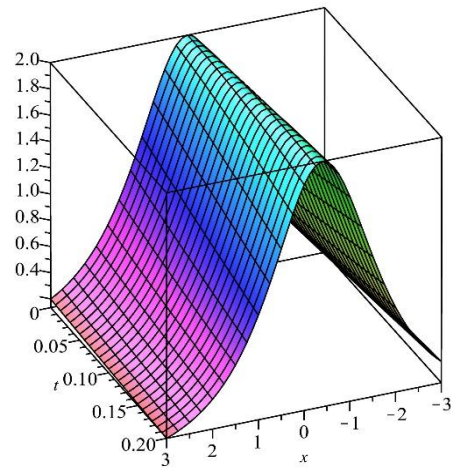


Fig. 2: Profile of the approximate solution $\tilde{v}_4(x,t)$ when $c=1$

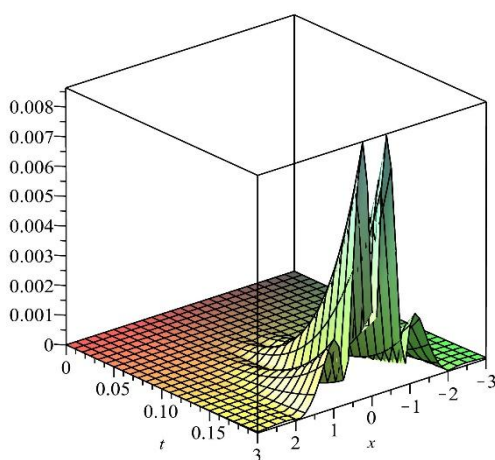


Fig. 3: Profile of the absolute error of $\tilde{u}_4(x,t)$ when $c=1$

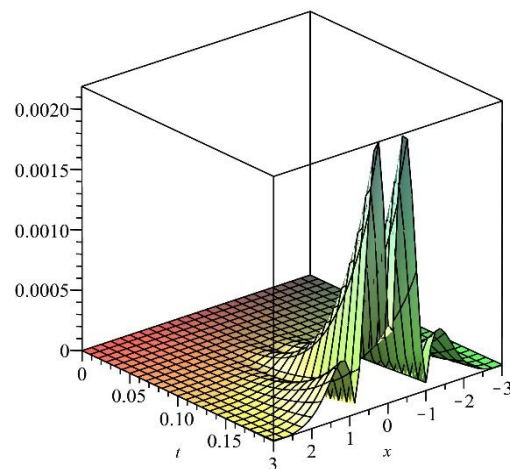


Fig. 4: Profile of the absolute error of $\tilde{v}_4(x,t)$ when $c=1$

To verify the veracity of the obtained solutions, the 4-terms approximate solutions along with their absolute errors are depicted in Figs. 1-4. From which, we can see that the approximate solutions have a good precision and the absolute errors start increasing in the nearby area of $x = 0$ as t increases. This could be interpreted from the behavior of the hyperbolic secant function that has exponentially decaying wings, which is represented in the soliton solutions (28). In addition, a comparison between the 4-terms approximate solutions and the exact solutions is presented in Tables 2 and 3. The numerical results reveal that the approximate solutions with small n are quite enough to give high accuracy results for small values of t . Moreover, it is noticeable that as x increases, the absolute errors become smaller.

Table 2: Comparison of the numerical results of $\tilde{u}_4(x,t)$ with the exact solution for $c = 1$

t	x	$u(x,t)$	$\tilde{u}_4(x,t)$	$ u(x,t) - \tilde{u}_4(x,t) $
0.05	-2	0.8287801526	0.8287801616	9.00000E-09
	-1	0.0261652778	0.02616305337	2.22440E-06
	0	-0.980132582	-0.9801333332	7.51000E-07
	1	0.2815973676	0.2815996627	2.29510E-06
	2	0.8835539226	0.8835538949	2.77000E-08
0.1	-2	0.7928832520	0.7928831806	7.14000E-08
	-1	-0.118110335	-0.1181792578	6.89230E-05
	0	-0.922085966	-0.9221333332	4.73670E-05
	1	0.3899600076	0.3900335963	7.35887E-05
	2	0.9041493116	0.9041481956	1.11602E-06
0.15	-2	0.7499802588	0.7499765076	3.75120E-06
	-1	-0.269479180	-0.2699783663	4.99186E-04
	0	-0.830273924	-0.8307999999	5.26076E-04
	1	0.4851336066	0.4856900046	5.56398E-04
	2	0.9211778920	0.9211679022	9.98978E-06

Table 3: Comparison of the numerical results of $\tilde{v}_4(x,t)$ with the exact solution for $c = 1$

t	x	$v(x,t)$	$\tilde{v}_4(x,t)$	$ v(x,t) - \tilde{v}_4(x,t) $
0.05	-2	0.5851834710	0.5851835896	1.18600E-07
	-1	1.395589282	1.395589494	2.12000 E-07
	0	1.990041498	1.990041666	1.68000 E-07
	1	1.198668121	1.198667839	2.82000 E-07
	2	0.4825890124	0.4825889053	1.07100 E-07
0.1	-2	0.6436097390	0.6436137459	4.00690 E-06
	-1	1.495399836	1.495405226	5.39000 E-06
	0	1.960655995	1.960666666	1.06710 E-05
	1	1.104572309	1.104562423	9.88600 E-06
	2	0.4378371578	0.4378339013	3.25650 E-06
0.15	-2	0.7071346990	0.7071667816	3.20826E-05
	-1	1.593410920	1.593439890	2.89700E-05
	0	1.913255824	1.913375000	1.19176E-04
	1	1.014757501	1.014676952	8.05490E-05
	2	0.3970443502	0.3970208250	2.35252E-05

4. Conclusion

In this work, the reduced differential transform method is effectively utilized to retrieve the approximate solution of a coupled KdV-type equations, namely the Kersten-Krasil'shchik coupled KdV-mKdV system. The solution is expressed as a convergent power series with simply computed components. This technique does not require additional operations such as linearization, discretization or perturbation. Therefore, its solution procedure is simpler than other existing methods. Our results presented in Figs. 1-4 and Tables 2 and 3 reveal the consistency, straightforwardness and robustness of the RDTM as an elegant mathematical tool for handling NLPDEs. The Kersten-Krasil'shchik coupled KdV-mKdV system may be investigated further by other existing approaches in forthcoming works.

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