Appell Type Changhee Polynomials Operational Matrix of Fractional Derivatives and its Applications

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Abstract: In this paper, a fractional order differential equation (FDEs), will be solved numerically through a new approximative technique based on Appell type Changhee polynomials. The operational of fractional order derivative will be constructed, then its application together with collocation method in solving fractional differential equations (FDEs) will be presented. The fractional derivatives in the FDEs are described in the Caputo sense. Some numerical examples are finally given to show the accuracy and applicability of the new operational matrix.

Keywords: Appell Type Changhee Polynomials, Operational Matrix, Collocation Methods

1. Introduction

The discovery of its numerous applications in the domains of mathematics, physics, biology, chemistry, engineering, and others has made fractional differential equations (FDEs) an extremely appealing research subject (Kilbas, Srivastava, & Trujillo 2006; Ray, Chaudhuri & Bera, 2006). Due to their nonlocal nature, it was already understood that modeling of physical processes that depended not only on the instant time but also on the preceding time history could be accomplished efficiently using calculus of arbitrary order. As a result, the numerical solutions of physical-interested FDEs have received a lot of attention. However, because most FDEs lack analytic solutions, numerical approaches that allow for quick and reliable evaluation of approximate solutions in a variety of ways are essential. In the literature, there are several methods for solving FDEs, see (Ray, Chaudhuri, & Bera 2006; Yang, Xiao, & Su, 2010; Odibat 2011). The operational matrices with arbitrary order derivative and integration have been found for a variety of orthogonal and non-orthogonal polynomials, such as Chebyshev polynomials, Legendre polynomials, Jacobi polynomials and Genocchi polynomials (Doha, Bhrawy, & Ezz-Eldien, 2011; Saadatmandi, & Dehghan, 2010; Doha, Bhrawy, & Ezz-Eldien, 2012; Isah & Phang, 2019; Isah & Phang, 2017), and many others. In this paper we will, for the first time, derive Appell type Changhee polynomials operational matrix of fractional derivative and apply it solve FDEs using the collocation approach.

The paper is organized as follows: the second section covers the necessary mathematical preliminaries for fractional calculus. Changhee and Appell type Changhee polynomials, some of their features, and

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arbitrary function approximation are covered in section three. In part four, we develop the Appell type Changhee polynomials operational matrix of fractional order derivative, and in section five, we explain how the collocation points are employed to solve FDEs using these matrices. We solve various numerical examples in section six. We come to a conclusion in section seven.

2. Preliminaries

2.1 Fractional Integration and Derivative

Many definitions for fractional order integration and differentiation are on the increase, but the most important are those utilized in the development of fractional calculus theory which are the Riemann – Liouville and Caputo fractional derivative definition, which are defined as follows:

Definition 2.1 The integral I due to Riemann and Liouville of fractional order α of $f(t)$ is given by (Kilbas, Srivastava, & Trujillo 2006).

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad t > 0, \alpha \in \mathbb{R}^{+} \qquad [1]$$

Where, $\Gamma(\cdot)$ is the gamma function. Below are properties of I^{α}

$$I^{\alpha} I^{\beta} f(t) = I^{\alpha+\beta} f(t), \quad \alpha > 0, \ \beta > 0 \qquad [2]$$

$$I^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} t^{\beta+\alpha} \qquad [3]$$

The integration is linear i.e

$$I^{\alpha}\big(\lambda f(x) + \mu g(x)\big) = \lambda I^{\alpha} f(x) + \mu I^{\alpha} g(x) \qquad [4]$$

With, λ and μ constants.

Definition 2: The derivative due to Riemann and Liouville of order α (D_L^{α}) of a function $f(t)$ is defined as

$$D_L^{\alpha} f(t) = \frac{d^m}{dt^m}\left(I^{m-\alpha} f(t)\right), \quad m-1 < \alpha \leq m, \ m \in \mathbb{N} \qquad [5]$$

When modeling some real-world issues, the Riemann-Liouville definition has some drawbacks (Doha, Bhrawy, & Ezz-Eldien, 2011; Saadatmandi, & Dehghan, 2010). However, the Caputo's definition was meant to address such issues, and we utilize it as described in the following definition

Definition 2.1 The fractional derivative D^{α} in Caputo sense of a function $f(x)$ is defined as in (Doha, Bhrawy, & Ezz-Eldien, 2011; Saadatmandi, & Dehghan, 2010) by:

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}} d\tau, \qquad n-1 < \alpha < n, \ n \epsilon \mathbb{N} \qquad [6]$$

The Caputo fractional derivative has the following properties

$$D^{\alpha} C = 0, \qquad where\ C\ is\ constant$$

$$D^{\alpha}x^{\sigma}=\frac{\Gamma(\sigma+1)}{\Gamma(\sigma+1-\alpha)}t^{\sigma-\alpha}, if \quad \sigma\in\mathbb{N}\cup\{0\} \ \ and \ \ \sigma\geq\lceil\alpha\rceil \qquad [7]$$

Where, $\lceil\alpha\rceil$ denote ceil function.

Caputo fractional operator also linear since

$$D^{\alpha}\big(\lambda f(t)+\mu g(t)\big)=\lambda D^{\alpha}f(t)+\mu D^{\alpha}g(t). \qquad [8]$$

with λ and μ constants.

2.2. Changhee Polynomials and Some Properties

It is well known that Changhee polynomials $Ch_n(x)$ and Changhee numbers Ch_n are defined usually using the generating functions (Kim, Kim, & Seo, 2013; Lee et al., 2016).

$$\frac{2}{t+2}(1-t)^{x}=\sum_{n=0}^{\infty}Ch_n(x)\frac{t^n}{n!} \qquad [9]$$

And when $x=0,\ Ch_n=Ch_n(0)$ are the Changhee numbers, see (Kim, Kim, & Seo, 2013)

The Changhee polynomials can also be obtained by

$$Ch_n(x)=\sum_{n=0}^{\infty}S_1(n,l)E_l(x)$$

Where, $S_1(n,l)$ and $E_l(x)$ are Sterling numbers of first kind and Euler polynomials respectively.

However, the Appell type Changhee polynomials $Ch_n^*(x)$ are defined by the generating function given by

$$\frac{2}{t+2}e^{xt}=\sum_{n=0}^{\infty}Ch_n^*(x)\frac{t^n}{n!} \qquad [10]$$

It was also established that the Changhee numbers $Ch_n^*=Ch_n^*(0)$ are equal to the $Ch_n=Ch_n(0)$ see (Lee et.al 2016). Thus, for $n\in\ \mathbb{N}$ we can obtain the n^{th} degree Appell type Changhee polynomials by;

$$Ch_n^*(x)=\sum_{m=0}^{n}\binom{n}{m}Ch_{n-m}^*x^m \qquad [11]$$

From equation (11) one can easily see that;

$$\frac{d}{dx}Ch_n^*(x)=\frac{d}{dx}nCh_{n-1}^*(x) \qquad [12]$$

And it is very clear from (12) we have

$$Ch_n^*(x)=\int_0^x nCh_{n-1}^*(s)ds+\ Ch_n^* \qquad [13]$$

It is also worth noting that $Ch_0^*=1$ and $2Ch_n^*+nCh_{n-1}^*=0,\ \forall\, n\geq 1$

One should also note the Appell type Changhee polynomials satisfy the identity (12)

$$\int_0^1 Ch_n^*(x)Ch_m^*(x)\,dx = \sum_{i=0}^{m}\sum_{k=0}^{m-i}\binom{m}{i}\frac{(-1)^{m-i-1}(m-i)\binom{m-i}{k}Ch_k^*(1)Ch_i^*}{(2(m-i)-k+1\binom{2(m-i)-k}{m-i})} \quad [14]$$

3. Function Approximation

Let $\{Ch_1^*(x), Ch_2^*(x), \ldots, Ch_n^*(x)\} \subset \mathrm{L}^2[0,1]$ be the set of Appell type Changhee polynomials and suppose that

$Y = Span\{ Ch_{\mathbf{1}}^*(x), Ch_{\mathbf{2}}^*(x), \ldots, Ch_n^*(x) \}$.

For $g(t)$ an arbitrary element of $\mathrm{L}^2[0,1]$, $g(t)$ has a good and unique approximation in Y, as Y is a finite dimensional subspace of $\mathrm{L}^2[0,1]$ see (Kreyszig, 1978). If, say, $g^*(t)$ is the unique approximation of $g(t)$ we can have

$$\forall\, y(t) \in Y, ||g(t) - g^*(t)||_2 \le \|g(t) - y(t)\|_2 \quad [15]$$

But because Y is a closed subspace of $\mathrm{L}^2[0,1]$, then according to (Kreyszig, 1978). $\mathrm{L}^2[0,1] = \mathrm{Y} \oplus \mathrm{Y}^\perp$, where, $\mathrm{Y}^\perp$ denote the orthogonal complement of Y, and so we have $g(t) = y(t) + s(t)$ and then $s(t) = g(t) - y(t)$, which also means that $g(t) - g^*(t) \in Y^\perp$. Therefore, this shows that $\forall\, y(t) \in Y$

$$\langle g(t) - g^*(t), y(t)\rangle = 0 \quad [16]$$

$\langle\, . \rangle$ denotes inner product.

Since$g^*(t) \in Y$, then there exists $c_1, c_2, \cdots, c_N$ such that

$$g(t) \approx g^*(t) = \sum_{i=0}^{N} c_i Ch_i^*(x) = C^T Ch^*(x) \quad [17]$$

Where

$$\boldsymbol{C} = [c_1, c_2, \cdots, c_N]^T,$$

$$Ch^*(x) = [Ch_1^*(x), Ch_2^*(x), \ldots, Ch_n^*(x)]^T \quad [18]$$

Using Equation (16) we have

$$\langle g(t) - C^T Ch^*(x), Ch_i^*(x)\rangle = 0$$

For simplicity we can write

$$\langle g(t), Ch^*(x)\rangle = C^T \langle Ch^*(x), Ch^*(x)\rangle \quad [19]$$

Where $\langle Ch^*(x), Ch^*(x)\rangle$ is an $N \times N$ matrix. Let

$$W = \langle\, Ch^{*}(x), Ch^{*}(x) \rangle = \int_{0}^{x} Ch^{*}(s), Ch^{*^{T}}(s)\ ds \qquad [20]$$

Thus W can be calculated by using (14). And, from (19) and (20) we get

$$C \ = \ W^{-1} \langle\, g(t), Ch^{*}(x) \rangle \qquad [21]$$

4. Appell Type Changhee Operational Matrix of Fractional Derivative

Consider the vector of Appell type Changhee $Ch^{*}(x)$ given in (18), then the Caputo fractional derivative of $Ch^{*}(\boldsymbol{x})$ can be written as

$$D^{\alpha} Ch^{*}(x) = \Psi^{(\alpha)} Ch^{*}(x) \qquad [22]$$

Where, $\Psi^{(\alpha)}$ denotes the operational matrix of arbitrary order derivative of dimension $N \times N$, which we will show how it is obtained the theorem below;

Theorem (Main)

Suppose $Ch^{*}(x)$ is the Appell type Changhee vector given in (17) and let $\alpha > 0$. Then, the operational matrix $\Psi^{(\alpha)}$ shown in (22) is given by

$$\boldsymbol{\Psi}^{(\boldsymbol{\alpha})} = \begin{bmatrix} \sigma_{1,1,1} & \sigma_{1,2,1} & \cdots & \sigma_{1,N,1} \\ \sum_{k=1}^{2} \sigma_{2,1,k} & \sum_{k=1}^{2} \sigma_{2,2,k} & \cdots & \sum_{k=1}^{2} \sigma_{2,N,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=1}^{m} \sigma_{m,1,k} & \sum_{k=1}^{m} \sigma_{m,2,k} & \cdots & \sum_{k=1}^{m} \sigma_{m,N,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=1}^{N} \sigma_{N,1,k} & \sum_{k=1}^{N} \sigma_{N,2,k} & \cdots & \sum_{k=1}^{N} \sigma_{N,N,k} \end{bmatrix}$$

Where, $\sigma_{m,n,k}$ is given by

$$\sigma_{m,n,k} = m! \frac{Ch_{m-k}^{*}}{(m-k)!\, \Gamma(k+1-\alpha)} c_{n,k} \qquad [23]$$

Ch_{n-1}^{*} is the Changhee number and $c_{n,k}$ can be obtained from (21).

Proof

Consider the Appell type Changhee polynomial $\mathrm{Ch}_{\mathrm{m}}^{*}(\mathrm{x})$ of degree m, with $\mathrm{m} = 1,2, \dots, \mathrm{N}$, and by using (3)(4) and (10) we can have

$$D^{\alpha}Ch_m^*(x) = \sum_{k=1}^{m} \frac{m!\,Ch_{m-k}^*}{(m-k)!\,k!} Dx^k = \sum_{k=1}^{m} \frac{m!\,Ch_{m-k}^*}{(m-k)!\,k!} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}\, x^{k-\alpha}$$
$$= \sum_{k=1}^{m} \frac{m!\,Ch_{m-k}^*}{(m-k)!\,\Gamma(k+1-\alpha)}\, x^{k-\alpha} \qquad [24]$$

Approximating the function $f(x) = x^{k-\alpha}$ using truncated Changhee series i.e.

$$f(x) = \sum_{n=0}^{N} c_{n,k} Ch_n^*(x) \qquad [25]$$

Subsituting (25) into (24) we get

$$D^{\alpha}Ch_m^*(x) = \sum_{n=1}^{N} \left(\sum_{k=1}^{m} \frac{m!\,Ch_{m-k}^*}{(m-k)!\,\Gamma(k+1-\alpha)}\, c_{n,k} \right) Ch_n^*(x)$$
$$= \sum_{n=1}^{N} \left(\sum_{k=1}^{m} \sigma_{m,n,k} \right) Ch_n^*(x) \qquad [26]$$

Where, $\sigma_{m,n,k}$ is given in (23), thus, putting (26) in vector form becomes

$$D^{\alpha}Ch_m^*(x) = \left(\sum_{k=1}^{m} \sigma_{m,1,k} \quad \sum_{k=1}^{m} \sigma_{m,1,k} \quad \ldots \quad \sum_{k=1}^{m} \sigma_{m,1,k} \right) \boldsymbol{Ch}^*(\boldsymbol{x}) \quad m = 1,2,\ldots,N \qquad [27]$$

This completes the proof.

5. Collocation Methods Using Appell Type Changhee Matrix of Fractional Integration

Consider the fractional multi-order differential equation of the form

$$D^{\alpha}y(x) = \sum_{j=1}^{n} a_j D^{\lambda_j}\, y(x) + a_{n+1}y(x) + g(x) \qquad [28]$$

Subject to

$$y^{(j)}(0) = d_j, \quad j = 0,1,\ldots,n-1 \qquad [29]$$

Where, $a_j, j = 0,1,\ldots,n+1$ are real constant coefficients and $n-1 < \alpha \le n, \quad 0 < \lambda_1 < \cdots < \lambda_n < \alpha$.

We first approximate $D^{\alpha}\mathrm{y}(x), D^{\lambda_j}\, y(x)\ and\ \ \mathrm{y}(x)$ by means of Appell type Changhee polynomials as in (17)

$$y(x) = \sum_{i=0}^{N} c_i Ch_i^*(x) \ = \ C^T Ch^*(x) \qquad [30]$$

where, $C^T Ch^*(x)$ are given in (18)

Employing (22) on (30) we obtain

$$D^{\alpha}\mathrm{y}(t) \simeq \mathrm{C}_j \Psi^{(\alpha)} Ch^*(x)^{\boldsymbol{T}}, \quad j = 1,2,\cdots,n. \qquad [31]$$

Similarly,

$$D^{\lambda_j} y(x)) \simeq \mathrm{C}_i \Psi^{(\lambda_\mathrm{j})} Ch^*(x)^{\boldsymbol{T}}, \quad i = 1,2,\cdots,n. \qquad [32]$$

Therefore, substituting (30), (31) and (32) in (28), we have

$$\mathrm{C}\Psi^{(\alpha)} Ch^*(x)^T = \sum_{i=1}^{n} a_i C\Psi^{(\lambda_\mathrm{j})} Ch^*(x)^T + a_{n+1} C^T Ch^*(x) + g(x) \qquad [33]$$

For the initial conditions in (29) we have

$$\mathrm{C}\Psi^{(\mathrm{j})} \mathrm{S}(1)^T = \gamma_j \quad . \qquad [34]$$

For the solution of (28), we put the collocation points $x_i = \frac{i}{N-1} + 1, \quad i = 1,2,\cdots,N-1$ on (33) and we get

$$\mathrm{C}\Psi^{(\alpha)} \mathrm{Ch}^*(\mathrm{x_i})^\mathrm{T} = \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{a_i} \mathrm{C}\Psi^{(\lambda_\mathrm{j})} \mathrm{Ch}^*(\mathrm{x_i})^\mathrm{T} + \mathrm{a}_{\mathrm{n}+1} \mathrm{C}^\mathrm{T} \mathrm{Ch}^*(x_i) + g(x),\ i = 1,2,\cdots,N-1 \qquad [35]$$

Thus, (35) are $n(N-1)$ equations in C_i . These equations and (34) make $n(N)$ equations which can be solved easily and consequently we can obtain $\mathrm{y}(x)$ given in (30).

6. Application on Numerical Example

We now consider some numerical examples and apply the method described. We used Maple 25 to carry out all the computations

Example 1.

Consider the equation

$$D^\alpha y(x) + y(x) = x^2 + 2x^{2-\alpha}, \quad n-1 < \alpha < n, \qquad 0 < x < 1$$

Subject to initial condition $y^{(k)}(0) = 0, k = 0,1,\ldots n-1$

This equation has an exact solution $y(x) = x^2$ when $\alpha = 1$, we used this method with $N = 6$ only, we obtained a result that fit very well with the exact solution as shown in figure1 below. Also we solve this problem when $\alpha = 0.9\ and\ 0.8$ in which we observed that as we move α close to 1 our solution moves also close the solution of the equation when $\alpha = 1$. This is shown in Figure 2.

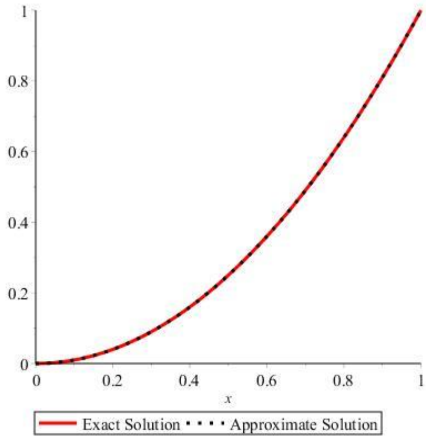

Figure 1: Comparison of exact and approximate solution for example 1

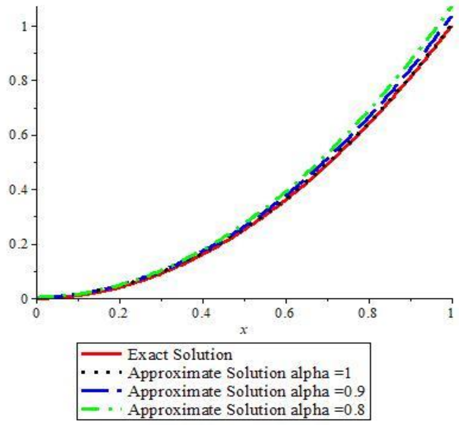

Figure 2: Comparison of Approximate and exact solution with the approximate solutions when $\alpha = 0.9$ *and* 0.8

Example 2.

We consider the following multi-term fractional differential equation which was solved in (Bhrawy, & Alofi, 2013)

$$D^2y(x) - 2Dy(x) + D^{\frac{1}{2}}y(x) + y(x) = f(x), \quad y(0) = 0, y'(0) = 0, \quad x \in [0,1].$$

Where, $f(x) = x^3 - 6x^2 + 6x + \frac{16}{5\sqrt{\pi}}x^{2.5}$.

$y(x) = x^3$ is the exact solution to this problem. We solved this equation with $N = 10$ using our method and compare the result to the exact answer illustrated in figure 3. This demonstrates that our solution is compatible with the exact solution.

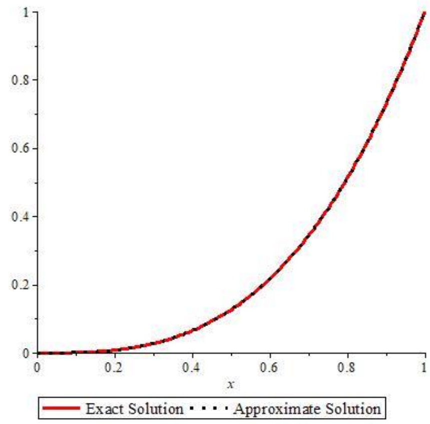

Figure 3: Exact and approximate solution comparison for example 2

7. Conclusion

We developed a new fractional order derivative matrix based on Appell type Changhee polynomials in this study, which is the first time Appell type Changhee polynomials have been used in the context of fractional differential equation solutions to our knowledge. In order to solve FDEs, this new operational matrix is used with the collocation points method. Our findings indicate that this new matrix is promising and can be utilized to solve a variety of difficult FDEs.

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