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A Posteriori $L_\infty(L_2)$ and $L_\infty(H^1)$ Error Analysis of Semdiscrete Semilinear Parabolic Problems

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Abstract: This paper aims to construct a posteriori error bounds for semilinear parabolic equations. The derivation of this bound is inspired by Makridakis and Nochetto 2003. Some challenges have been addressed through Lipschitz conditions and Gronwall's inequality. The curtail idea for proving these estimators is to reduce the computation of schemes.

Keywords: Posteriori Error Estimate, Semilinear Parabolic Problems, Finite Element Methods

1. Introduction

The finite element (FEM) and compact finite difference (CFD) methods are one of the most flexibility common techniques used for solving partial differential equations for various kinds of application in many fields, for instance, in engineering, chemistry and biology (Ainsworth & Oden, 2000; Akrivis, Makridakis, & Nochetto, 2009; Manaa, Moheemmeed, & Hussien, 2010; Hussein, 2011; Sabawi, Pirdawood, & Khalaf, 2021; Sabawi, Pirdawood, & Rasool, 2021; Sabawi, Pirdawood, & Sadeeq 2021; Pirdawood & Sabawi, (2021). A posteriori error estimation for linear paraboilc problems have been extensively studied for the past three decades. These errors play a crucial rule in reducing computational scheme. Many researchers have derived a posteriori error estimation for linear parabolic problems (Makridakis & Nochetto, 2003; Lakkis & Makridakis, 2006; Makridakis, 2007; Bänsch, Karakatsani & Makridakis, 2012; Demlow, Lakkis, & Makridakis, 2009; Kopteva & Linss, 2013).

The derivation of above works is inspired by Makridakis and Nochetto (2003) and then extended for fully discrete approximation by Lakkis and Makridakis (2006). Recently, Sabawi has constructed a posteriori error estimate for a class of semilinear parabolic problems with nonlipschitz continuity case, for more details see, Sabawi (2019), Sabawi (2020), Sabawi (2021), Sabawi (2021). The main technique used in this work is based on the Makridakis and Nochetto (2003). The main contribution of this work is to extend the case in Makridakis and Nochetto (2003) to the case of semilinear problem with lipschitz condition. The rest of this paper is structured as follows. In Section 2, the model problem is introduced and finite element Galerkin method with some necessary background results is discussed.

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Section 3, $L_\infty(L_2)$ and $L_\infty(H^1)$ error bounds for such problems, are proved. Conclusions are given in Section 4.

2. Model Problem and Notation

Consider the model semilinear parabolic problem

$$\begin{aligned} \frac{\partial u}{\partial t} + \Delta u &= f(u),\ (x,t) \in \Omega \times, \\ u(x,t) &= 0, (x,t) \in \partial\Omega \times, \\ u(x,0) &= u_0(x), x \in \Omega, \end{aligned} \qquad [1]$$

where Ω is a plane convex domain subset of $\mathbb{R}^k, \Omega \subset \mathbb{R}^k$ with smooth boundary condition $\partial\Omega$, where $u_t = \partial u/\, \partial t = \partial_t u, \Delta u = \sum_{i=1}^{d} \partial^2 u/\, \partial x_i^2$. Let the f nonlinear forcing term satisfy the following lipschitz condition.

$$\begin{aligned} &|f(z_1) - f(z_2)| \\ &\le C|z_1 - z_2|. \end{aligned} \qquad [2]$$

Next, we have

$$\| v \|_{L_p(0,T;X)} = \left(\int_0^T \| v(t) \|_x^p \, dt \right)^{1/p} \infty,\ 1 \le p < \infty$$

and

$$\| v \|_{L_\infty(0,T;X)} = \text{esssup} \| v(t) \|_X < +\infty,\ p = +\infty.$$

We define the norm $L_2(\Omega)$, which denoted by $\|.\|$ on Ω by L_2-inner product $\langle .,. \rangle$, and also, define the Hilbert space $H = L_2(\Omega)$ with standard norm and inner product

$$\| v \| = \left(\int_\Omega |v|^2 \, \mathrm{d}x \right)^{1/2},\ \langle v, w \rangle = \int_\Omega vw \, \mathrm{d}x.$$

$v \in H_0^1(\Omega)$. Multiplying (1) by test function $v \in H_0^1(\Omega)$ and integrating over the domain Ω, implies

$$\begin{aligned} &\int_\Omega \partial_t uv dx - \int_\Omega \Delta uv dx \\ &= \int_\Omega f(u) v \, \mathrm{d}x. \end{aligned} \qquad [3]$$

Applying the divergence theorem on the second term on the left hand side, gives

$$\begin{aligned} &\int_\Omega \partial_t uv \, \mathrm{d}x + \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x \\ &= \int_\Omega f(u) v \, \mathrm{d}x. \end{aligned} \qquad [4]$$

Above equation can be transformed to the weak form problem. Find $\mathrm{u} \in \mathrm{H}_0^1(\Omega)$, becomes

$$\langle \partial_t u, v\rangle + a(u,v) = \langle f(u), v\rangle,\ \forall v \in H_0^1(\Omega), \qquad [5]$$

where

$$\langle \partial_t u, v\rangle = \int_\Omega \partial_t u v dx,$$
$$a(u,v) = \int_\Omega \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_j}\,\mathrm{d}x,$$
$$\langle f(u), v\rangle = \int_\Omega f(u) v\,\mathrm{d}x.$$

To process with finite element approximation. We restrict $S_h \subset H_0^1$, and define the approximate solution by $u_h: [0,T] \to S_h$ such that

$$\langle \partial_t u_h, \chi\rangle + a(u_h, \chi) = \langle f(u_h), \chi\rangle,\ \ \chi \in S_h, \qquad [6]$$

with $u_h(0) = v_h$ where $v_h \in S_h$ is approximation of v. Let

$$S_h = \{\chi \in H_\circ^1(\Omega): \chi|_K \in P_k(K)\ \forall K \in T_h\},$$

where T_h is a member of a family of quasiuniform partition on Ω and $P_k(K)$ is the space of polynomial of degree k over K.

3. $L_\infty(L_2)$ and $L_\infty(H^1)$ a Posteriori Error Bounds

We require some a posteriori error bounds for a related elliptic problem in the forthcoming error analysis. To that end, we introduce elliptic reconstruction to be unique operator defined as $R: S_h \longrightarrow H_0^1(\Omega)$ such that

$$a(w(t), v) = \langle g_h(t), v\rangle = a(u_h, v), \qquad [7]$$

for all $v \in H_\circ^1(\Omega)$,, where $g_h(t) = A_h u_h - P_h^\circ f(u_h) + f(u_h)$, and $P_h^\circ: L_2(\Omega) \longrightarrow S_h$ is the L_2-projection so that

$$\langle P_h^\circ f(u), \chi\rangle = \langle f(u), \chi\rangle\ \forall \chi \in S_h, u \in H^1(\Omega), \qquad [8]$$

where $A_h: S_h \longrightarrow S_h$ is linear operator given by

$$a(u, \chi) = \langle A_h u, \chi\rangle\ \forall \chi \in S_h. \qquad [9]$$

Lemma 3.1. Given some $g \in L^2(\Omega)$, let $w \in H_0^1(\Omega)$ be the solution of the elliptic problem $a(w, v) = \langle g_b, v\rangle\ \forall v \in H_0^1(\Omega)$.

and let $u_h \in S_h$ be its cG approximation given by

$$a(u_h, v_h) = \langle g_h, v_h\rangle\ \forall v_h \in S_h,$$

then the following elliptic a posteriori bounds hold:

$$\begin{aligned} \|w-u_h\|_{L_2} &\leq C\mathcal{E}_{0,2}(w_h, g_h, T_h) \\ \|w-u_h\|_{H^{-1}} &\leq C\mathcal{E}_{-1,2}(w_h, g_h; T_h) \\ \|(w-u_h)\|_{H^1} &\leq C\mathcal{E}_{1,2}(w_h, g_h; T_h). \end{aligned}$$

Where

$$\begin{aligned} \mathcal{E}_{0,2}(u_h, g_h, T_h) &:= \left(\sum_{K\in T_h} h_K^4 \parallel R \parallel_{L_2(K)}^2 + \sum_{\varepsilon\in\varepsilon_h} h_\varepsilon^3 \|J(u_h)\|_{L_2(\varepsilon)}^2 \right) \\ \mathcal{E}_{-1,2}(u_h, g_h, T_h) &:= \left(\sum_{K\in T_h} h_K^6 \|\partial_t R\|_{L_2(K)} + \sum_{\varepsilon\in\varepsilon_h} h_\varepsilon^5 \|\partial_t J(u_h)\|_{L_2(\varepsilon)}^2 \right) \\ \mathcal{E}_{1,2}(u_h, g_h, T_h) &:= \left(\sum_{K\in T_h} h_K^2 \parallel R \parallel_{L_2(K)}^2 + \sum_{\varepsilon\in\varepsilon_h} h_\varepsilon \|J(u_h)\|_{L_2(\varepsilon)}^2 \right). \end{aligned}$$

Here, C is a positive constant that is independent of the data, the exact solution, the numerical solution and the meshsize but may be dependent upon the size of the domain Ω. Proof. To start with, $\|\partial_t(u_h - w)\|_{H^{-1}}$ for simplicity, suppose that $A = -\Delta$, where A is self-adjoint and positive definition.

$$\|\partial_t(w-u_h)\|_{H^{-1}} = \sup_{\|\chi\|_V=1} \langle \partial_t(w-u_h), \chi \rangle,$$

where $\chi \in H_0^1$ and $\partial_t\theta = \partial_t(w-u_h) \in V^{-1}$. Applying duality argument such that

$$a(\psi, v) = \langle v, \chi \rangle,\ \forall v \in H_0^1.$$

Assume that, there exists a constant $C_1 > 0$ which depend on Ω, gives

$$\parallel \psi \parallel_{H^3} \leq C_1 \parallel \chi \parallel_{H^1}.$$

If $T_h = \{K\}$ is is a shape-regular partition of Ω in to finite elements K, and let $\varepsilon_h = \{\varepsilon\}$ is the set of internal side, then we have

$$\begin{aligned} \langle \partial_t\theta, \chi \rangle = a(\partial_t\theta, \psi) &= a(\partial_t(w-u_h), \psi - I_h\psi) \\ &= a(\partial_t w, \psi - I_h\psi) - a(\partial_t u_h, \psi - I_h\psi) \\ &\leq C_I \sum_{K\in T_h} \mid \Delta\big(\partial_t(w-u_h)\big), \psi - I_h\psi \Big) \mid \\ &+ C_I \sum_{\varepsilon\in\varepsilon_h} \int_\varepsilon \mid [\partial_n(\partial_t(w-u_h)], \psi - I_h\psi \mid \qquad [10] \\ &\leq C_I \sum_{K\in T_h} \|\Delta(\partial_t\theta)\| \|\psi - I_h\psi\| \\ &+ C_I \sum_{\varepsilon\in\varepsilon_h} \int_\varepsilon \|[\partial_n(\partial_t\theta)]\| \|\psi - I_h\psi\|. \end{aligned}$$

Since

$$\begin{aligned} |\psi - I_h\psi|_{L_2(\Omega)} &\le Ch_K^3|\psi|_{H^3},\\ |\psi - I_h\psi|_{L_2(\partial\Omega)} &\le Ch_\varepsilon^{5/2}|\psi|_{H^3}, \end{aligned}$$

where $I_h \colon V \longrightarrow S_h$ is local interpolation operator and $[\partial_n\theta]$ is the spatial jump of the field $\nabla\theta$ across an element side e defined as

$$[\partial_n\theta] = \lim_{\varepsilon\to 0}[\nabla\theta(x+\varepsilon n_e) - \nabla\theta(x-\varepsilon n_e)],$$

where n_e is normal vector to e at the point x. If putting these equations in (10), this shows

$$\begin{aligned} &\eta_{-1}(\partial_t u_h)^2\\ &\le C_I \sum_{K\in T_h} h_K^6 \|\Delta(\partial_t\theta)\|^2 + C_I \sum_{\varepsilon\in\varepsilon_h} h_\varepsilon^5 \|[\partial_n(\partial_t u_h)]\|_{L_2(\varepsilon)}^2, \end{aligned} \qquad [11]$$

where $C_I > 0$ is an interpolation constant associated with local interpolation operator I_h. Now, let us recall elliptic reconstruction and semidiscrete scheme to simplify above equation, gives

$$\Delta(\partial_t\theta) = \Delta(\partial_t w) - \Delta(\partial_t u_h) = -A_h\,\partial_t u_h + P_h^\circ f'(u_h)\,\partial_t u_h - f'(u_h)\,\partial_t u_h - \Delta\,\partial_t u_h$$

since

$$\partial_t^2 u_h + A_h\,\partial_t u_h = P_h^\circ f'(u_h)\,\partial_t u_h.$$

Substituting this value in the above equation, we arrive

$$\begin{aligned} -P_h^\circ f'(u_h)\,\partial_t u_h + \partial_t^2 u_h \quad &+P_h^\circ f'(u_h)\,\partial_t u_h - f'(u_h)\,\partial_t u_h - \Delta\,\partial_t u_h\\ &= \partial_t[\partial_t u_h - \Delta u_h - f(u_h)]\\ &= \partial_t R, \end{aligned}$$

and

$$R|_K = \partial_t u_h - \Delta u_h - f(u_h)\; K\in T_h,\; J|_\varepsilon = [\partial_n u_h]\; \forall\varepsilon\in\varepsilon_h,$$

where $R|_K$ is the residual value and $J|_\varepsilon$ is the jump value . Substituting these values in (11), gives

$$\eta_{-1}(\partial_t u_h(t))^2 \le C_I \sum_{K\in T_h} h_K^6 \|\partial_t R\|_{L_2(\Omega)}^2 + C_I \sum_{\varepsilon\in\varepsilon_h} h_\varepsilon^5 \|J_t(u_h)\|_{L_2(\varepsilon)}^2$$

and we conclude

$$\|\partial_t(u-u_h)\|_{H^{-1}} \le C_I C_1 \eta_{-1}(\partial_t u_h(t)),\; k\ge 2$$

Similarly, to estimate $\|w-u_h\|_{L_2(\Omega)}$. This follows as

$$a\big((w-u_h),\Theta - I_h\Theta\big) \le C_I \sum_{K\in T_h} |\Delta(w-u_h),\Theta - I_h\Theta| + \sum_{\varepsilon\in\varepsilon_h} |J(u_h),\Theta - I_h\Theta|. \qquad [12]$$

Using (7) and semidiscrete scheme, gives

$$\begin{aligned}\Delta\theta = \Delta w - \Delta u_h \quad &= -A_h u_h + P_h^{\circ} f(u_h) - f(u_h) - \Delta u_h \\ &= -P_h^{\circ} f(u_h) + \partial u_h + P_h^{\circ} f(u_h) - f(u_h) - \Delta u_h \\ &= \partial_t u_h - \Delta u_h - f(u_h) \\ &= R,\end{aligned}$$

where R is the residual value. Since

$$|\Theta - I_h\Theta|_{L_2(\Omega)} \leq C h_K^2 |\Theta|_{H^2}$$
$$|\Theta - I_h\Theta|_{L_2(\partial\Omega)} \leq C h_\varepsilon^{3/2} |\Theta|_{H^2}.$$

Putting all together, reads

$$\eta_0(u_h)^2 = \sum_{K\in T_h} h_K^4 \parallel R \parallel_{L_2(K)}^2 + \sum_{\varepsilon\in\varepsilon_h} h_\varepsilon^3 \|J(u_h)\|_{L_2(\varepsilon)}^2.$$

From above equation, we have $\parallel \theta \parallel_{L_2} \leq C_I C_\Omega \eta_0(u_h(t)),\ k \geq 2$. Finally, to bound $\|(w - u_h)\|_{H^1}$. To achieve this, apply

$$\|\nabla(w - u_h)\| \leq \sup_{\|v\|=1} a(w - u_h, v),\ v \in H_0^1,$$

and $\theta = w - u_h$. From the above definition, this gives

$$\begin{aligned}a(\theta, v - I_h v) \quad &\leq \sum_{K\in T_h} \parallel \Delta\theta \parallel_{L_2(K)} \parallel v - Iv \parallel_{L_2(K)} \\ &+ \sum_{\varepsilon\in\varepsilon_h} \|J(u_h)\|_{L_2(\partial K)} \|v - I_h v\|_{L_2(\partial K)}.\end{aligned}$$

Where $\tilde{K}$ is the sub domain which contain of elements sharing a common edge with element K such that $\tilde{K} = \operatorname{int}\{\dot{K} \in \rho : \bar{K} \cap \hat{K} \neq \phi\}$. Then there exist a constant C which dependent of v and h_K, we have

$$\|v - I_h v\|_{L_2(K)} \leq C h_K \parallel v \parallel_{H^1},$$
$$\|v - I_h v\|_{L_2(\partial K)} \leq C h_\varepsilon^{1/2} \parallel v \parallel_{H^1},$$

where h_K is the diameter of the element K Ainsworth and Oden (2000). By using the definition of the reconstruction and the spatially method, yields

$$\begin{aligned}\Delta\theta \quad &= \Delta w - \Delta u_h \\ &= -A_h u_h + P_h^{\circ} f(u_h) - f(u_h) - \Delta u_h \\ &= -P_h^{\circ} f(u_h) + \partial_t u_h + P_h^{\circ} f(u_h) - f(u_h) - \Delta u_h \\ &= \partial_t u_h - \Delta u_h - f(u_h) \\ &= R,\end{aligned}$$

where R is the residual value. Putting all of these together, implies

$$\parallel \nabla\theta \parallel_V \leq C_I C_\Omega \eta_1(u_h(t)),\ k \geq 2,$$

where

$$\eta_1(u_h(t)) = \sum_{K\in T_h} h_K^2 \parallel R \parallel_{L_2(K)}^2 + \sum_{\varepsilon\in\varepsilon_h} h_\varepsilon \|J(u_h)\|_{L_2(\varepsilon)}^2$$

Lemma 3.2. (Main semilinear parabolic error). Let u and u_h be exact and approximate solutions defined in (1) and (6) respectively, and let w be elliptic reconstruction, the following error holds

$$\langle \partial_t\rho, v\rangle + a(\rho, v) = \langle f(u_h) - f(u), v\rangle + \langle \partial_t\theta, v\rangle. \qquad [13]$$

We shall split the error as follows:

$$e = u_h - u = w - u + u_h - w = \rho(t) + \theta(t),$$

where $w - u := \rho$ is the semilinear parabolic error, and $u_h - w := \theta$ is the elliptic error which can be estimated for elliptic problems.

Proof. By recalling (6) and (7), reads

$$\langle \partial_t u_h, v\rangle + a(w, v) = \langle f(u_h), v\rangle.$$

Subtracting (5) from this equation, we obtain

$$\langle \partial_t u_h - \partial_t u, v\rangle + a(w - u, v) = \langle f(u_h) - f(u), v\rangle.$$

Using elliptic reconstruction, gives

$$\begin{aligned}
\langle \partial_t e, v\rangle + a(\rho, v) &= \langle \partial_t u_h - \partial_t u, v\rangle + a(w - u, v)\\
&= \langle \partial_t u_h, v\rangle - \langle \partial_t u, v\rangle + a(w, v) - a(u, v)\\
&= \langle \partial_t u_h, v\rangle + a(w, v) - \langle \partial_t u, v\rangle - a(u, v)\\
&= \langle \partial_t u_h, v\rangle + a(w, v) - \langle f(u), v\rangle\\
&= (\partial_t u_h, v) + \langle A_h u_h - P_h^{\circ}(u_h) + f(u_h), v\rangle - \langle f(u), v\rangle\\
&= \langle f(u_h) - f(u), v\rangle.
\end{aligned}$$

Theorem 3.3. $L_\infty(H^1)$. Let u is exact solution and u_h is approximate solution, and function f satisfies (2). The following error estimate holds

$$\begin{aligned}
\max_{0\le t\le T} \|u - u_h\|_{L_\infty(0,T;H^1)}^2 &\le \|\nabla(Ru_h - u(0))\| + \max_{0\le t\le T} \parallel \nabla\theta \parallel\\
&+ \mathrm{e}^{4C_pC_fT}\left(\parallel \nabla\rho(0) \parallel^2 + 4C_fC_p \parallel \nabla\theta \parallel^2 + 2\|\partial_t\theta\|^2\right).
\end{aligned}$$

Proof. The error decomposed as

$$\|\nabla(u_h - u)\| \le \parallel \nabla\theta \parallel + \parallel \nabla\rho \parallel. \qquad [14]$$

To bound the first term on the right hand side of (14), we will use Lemma 3.1. The second term on the right hand side of (14) will be estimated by testing $v = \partial_t\rho$ in (13), implies

$$\langle \partial_t\rho, \partial_t\rho\rangle + a(\rho, \partial_t\rho) = \langle f(u) - f(u_h). \partial_t\rho\rangle + \langle \partial_t\theta, \partial_t\rho\rangle.$$

Applying Cauchy Schwarz inequality, gives

$$\|\partial_t\rho\|^2+\frac{1}{2}\frac{d}{dt}\parallel\nabla\rho\parallel^2\leq\|f(u)-f(u_h)\|\|\partial_t\rho\|+\|\partial_t\theta\|\|\partial_t\rho\|.$$

Using Poincare and Young's inequalities $\left(ab\leq\frac{\varepsilon}{2}a^2+\frac{1}{2\varepsilon}b^2\right)$, implies

$$\|\partial_t\rho\|^2+\frac{1}{2}\frac{d}{dt}\parallel\nabla\rho\parallel^2\leq C_fC_p\varepsilon[\parallel\nabla\rho\parallel^2+\parallel\nabla\theta\parallel^2]+\frac{1}{2\varepsilon}\|\partial_t\rho\|^2+\frac{\varepsilon}{2}\|\partial_t\theta\|^2+\frac{1}{2\varepsilon}\|\partial_t\rho\|^2.$$

After simplifying above equation, this gives us

$$\frac{\varepsilon-1}{\varepsilon}\|\partial_t\rho\|^2+\frac{1}{2}\frac{d}{dt}\parallel\nabla\rho\parallel^2\leq C_fC_p\varepsilon[\parallel\nabla\rho\parallel^2+\parallel\nabla\theta\parallel^2]+\frac{\varepsilon}{2}\|\partial_t\theta\|^2.$$

Picking $\varepsilon=2$, multiplying by 2 and taking integral from two sides, implies

$$\begin{aligned}\parallel\nabla\rho(t)\parallel^2+\int_0^t\|\partial_t\rho\|^2 &\leq\parallel\nabla\rho(0)\parallel^2+4C_fC_p\int_0^t\parallel\nabla\rho\parallel^2\\ &+4C_fC_p\int^t\parallel\parallel\nabla\theta\parallel^2+2\int_0^t\|\partial_t\theta\|^2\,\mathrm{d}s.\end{aligned}$$

Using Gronwall's lemma, the proof will be finished.

Theorem 3.4. $L_\infty(L_2)$. Let f satisfy (2). Then, the following estimate holds

$$\begin{aligned}\max_{0\leq t\leq T}\|u-u_h\|^2_{L_\infty(0,T;L_2)} &\leq C\|(Ru_h-u(0))\|+\max_{0\leq t\leq T}\parallel\theta\parallel\\ &+\left(\parallel\rho(0)\parallel^2+4C_f\parallel\theta\parallel^2+4C_fC_p\|\partial_t\theta\|^2\right)\mathrm{e}^{4C_fC_pT}.\end{aligned}$$

Proof. we shall split the error as

$$\|u_h-u\|\leq\parallel\theta\parallel+\parallel\rho\parallel. \qquad [15]$$

To handle with the first term on the right-hand side of (15), we will use Lemma 3.1 and, the second term on the right hand side will be bounded by setting $v=\rho$ in (13), implies

Proof.

$$\langle\partial_t\rho,\rho\rangle+a(\rho,\rho)=\langle f(u)-f(u_h),\rho\rangle+\langle\partial_t\theta,\rho\rangle.$$

Applying $ab\leq\frac{\varepsilon}{2}a^2+\frac{1}{2\varepsilon}b^2$, $\parallel\rho\parallel\leq C_p\parallel\nabla\rho\parallel$, and $|f(u)-f(u_h)|\leq C_f[\parallel\rho\parallel+\parallel\theta\parallel]$, implies

$$\frac{1}{2}\frac{d}{dt}\parallel\rho\parallel^2+\parallel\nabla\rho\parallel^2\leq C_fC_p\varepsilon[\parallel\rho\parallel+\parallel\theta\parallel]+\frac{1}{2\varepsilon}\parallel\nabla\rho\parallel^2+\frac{1}{2\varepsilon}\parallel\nabla p\parallel^2+C_p\varepsilon\|\partial_t\theta\|.$$

After simplifying above equation, yields

$$\frac{1}{2}\frac{d}{dt}\parallel\rho\parallel^2+\frac{\varepsilon-1}{\varepsilon}\parallel\nabla\rho\parallel^2\leq C_fC_p\varepsilon[\parallel\rho\parallel^2+\parallel\theta\parallel^2]+C_p\varepsilon\|\partial_t\theta\|^2.$$

If choosing $\varepsilon=2$, multiplying above equation by (2), and taking integral, this shows

$$\max_{0\le t\le T} \| \rho(t) \|^2 + \int_0^t \| \nabla\rho \|^2 \ \mathrm{d}t \le \| \rho(0) \|^2 + 4C_f C_p \| \rho \|^2 + \int_0^t \left(4C_f C_p \| \theta \|^2 + 4C_p \|\partial_t \theta\|^2\right) \mathrm{d}s.$$

4. Conclusion

The paper is concerning with derivation an optimal order a posteriori error estimates in term of the $L_\infty(L_2)$ and $L_\infty(H^1)$ norms for semidiscrete semilinear parabolic problems in the case when $f(u)$ Lipschitz are proved. The crucial tools in proving thiserror is reconstruction techniques introduced in 2003. This is consequently enabling us to use a posteriori error estimators derived for elliptic equation to obtain optimal order in terms of $L_\infty(L_2)$ and $L_\infty(H^1)$ norms for Lipschitz nonlinearities. Some challenges have to be overcome due to non-linearity on the forcing term depending on Gronwall's Lemma and Sobolev embedding. In the future, this paper can be extended to the fully discrete case for semilinear parabolic interface problems in $L_\infty(L_2) + L_2(H^1)$ and $L_\infty(L_2)$ norms, Sabawi (2017), Cangiani et al. (2018, 2020); Khalaf, Zeb, Sabawi, Diilali, and Wang (2021).

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